A physicist-friendly reformulation of the Atiyah-Patodi-Singer index (on a lattice)

M. F. Atiyah
(1929-2019)

* photo from Wikipedia

V. K. Patodi
(1945-1976)

* photo from mathshistory.st-andrews.ac.uk

I. M. Singer
(1924-2021)

* photo from Wikipedia

Hidenori Fukaya (Osaka U.)

F, Onogi, Yamaguchi, PRD96(2017) no.12, 125004 [arXiv: 1710.03379],
F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, *Commun.Math.Phys.* 380 (2020) 3, 1295-1311 [arXiv:1910.01987],
F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, PTEP 2020 (2020) 4, 043B04 [arXiv:1910.09675].
F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita, arXiv:2012.03543.
F, IJMPA 36 (2021) 26, 2130015 [2109.11147]

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Atiyah-Singer index theorem [1968] on a manifold without boundary

$$D\psi = 0 \qquad D := \gamma^\mu(\partial_\mu + iA_\mu)$$

Index theorem

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality #sol with - chirality

This text-book level theorem is physicist-friendly.

Atiyah-Patodi-Singer (APS) index theorem [1975]

is not very known

$$Ind(D_{\rm APS}) = \frac{1}{32\pi^2}\int_X d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD_Y)}{2}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

curvature

λ : eigenvalues of H

boundary Y

(because we were not very interested in manifolds with boundary until very recently).

APS index in topological insulator

Witten 2015 : APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator:

gapped material in the bulk but conductor on boundary (edge).

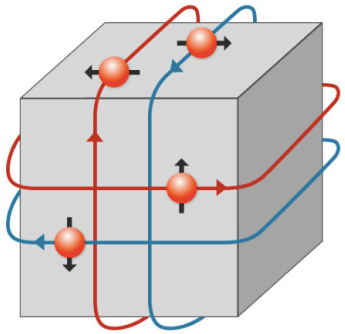
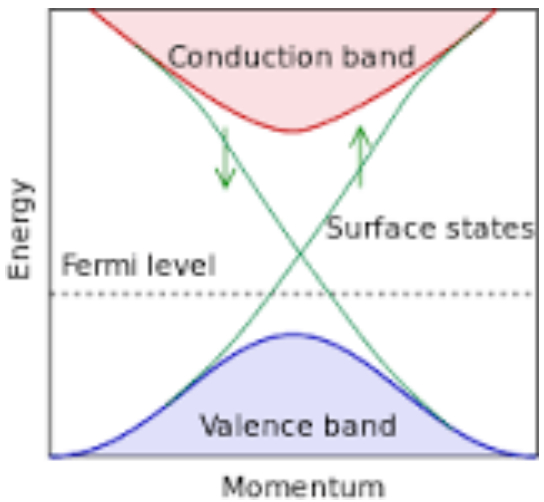


Figure from Wikipedia



2005 predicted by Kane et al.

2007 discovered [Koenig et al.].

T anomaly cancellation

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16&18, Freed-Hopkins 16, Witten 16, Yonekura 16&19, Witten-Yonekura 19…]

The APS index protects the Time reversal (T) symmetry.

fermion path integrals

$$Z_{\rm edge} \propto \exp(-i\pi\eta(iD^{\rm 3D})/2)$$ T-anomalous

$$Z_{\rm bulk} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$ T-anomalous

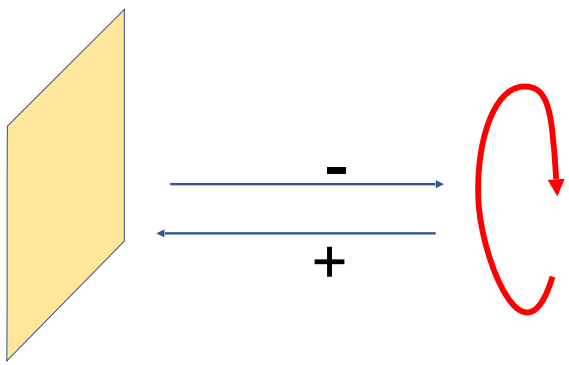
$$Z_{\rm edge}Z_{\rm bulk} \propto (-1)^{\mathfrak{I}} = (-1)^{-\mathfrak{I}}$$ ➡ T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

But the LHS $\mathfrak{I} = {\rm Ind}D_{\rm APS}$ of massless Dirac with non-local boundary condition is physicist-unfriendly.

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



angular momentum is conserved but chirality is not.

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition

[Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry to keep the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$\Rightarrow \quad \text{index} = n_+ - n_-$$

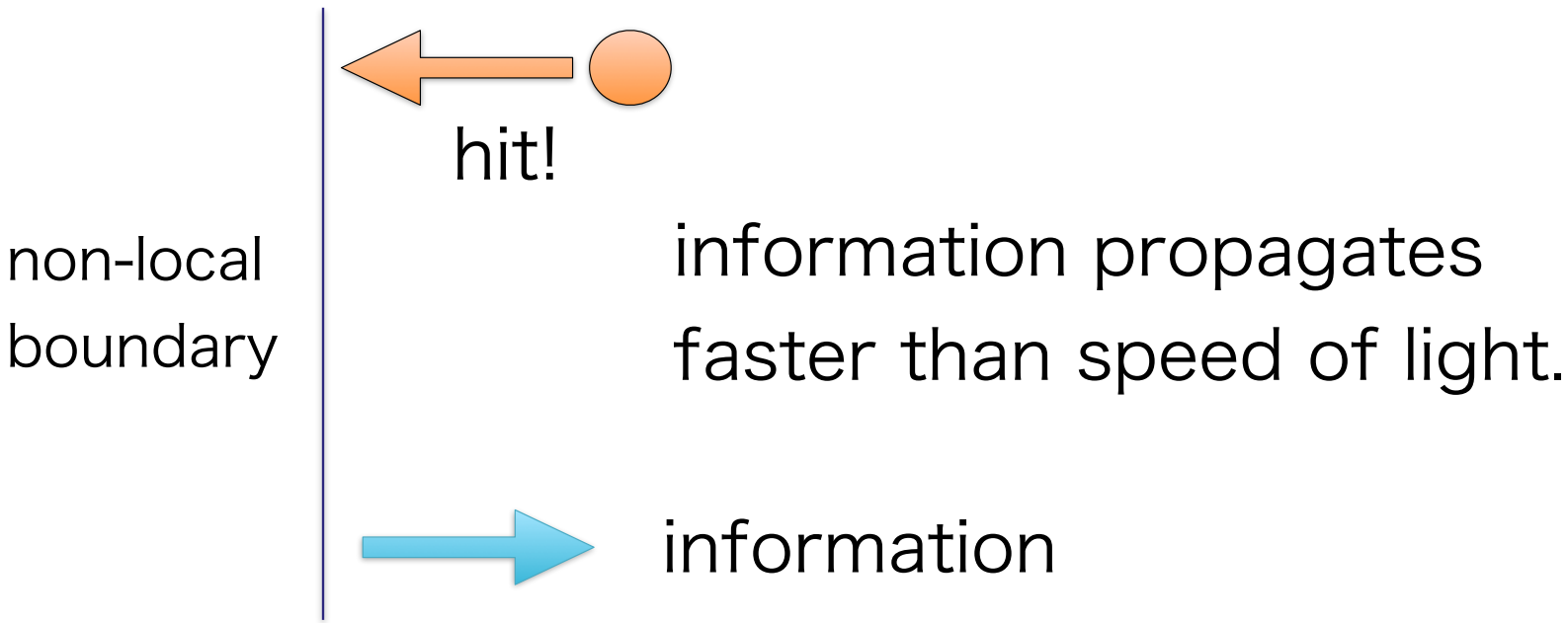
Mathematically beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.



Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.

→ need to give up chirality and consider L/R mixing

(massive case)

$$\cancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is “Yes, we can”.

References

In 2017, we proposed “A physicist-friendly reformulation of the Atiyah-Patodi-Singer index”.

[F, Onogi, Yamaguchi, arXiv:1710.03379]

In 2018, 3 mathematicians joined and we succeeded in a mathematical proof.

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, arXiv:1910.01987]

Lattice version:

[F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675]

Curved lattice version, F and S.Aoki, in progress.

Mod-two version (in odd dimensions):

[F, Furuta, Matsuki, Matuso, Onogi, Yamaguchi, Yamashita, arXiv:2012.03543]

A physicist-friendly review paper [H. Fukaya, IJMPA 36 (2021) 26, 2130015 arXiv: 2109.11147].

Contents

Massive Dirac fermion

Let us consider a Dirac fermion with negative mass (compared to regulator),

$$\frac{\det(D+m)}{\det(D-M)} \longrightarrow \quad \text{Pauli-Villars}$$

with SU(N) gauge fields on an even-dimensional closed flat Euclidean manifold.

Axial U(1) rotation

In the large mass limit, $m \to M \gg 0,$ let us perform an axial U(1) rotation with angle π,

$$M\bar{\psi}\psi \to M\bar{\psi}e^{\frac{i\pi}{2}\gamma_5}e^{\frac{i\pi}{2}\gamma_5}\psi = -M\bar{\psi}\psi$$

to flip the sign of mass.

$$\frac{\det(D+M)}{\det(D-M)} = \frac{\det(D-M)}{\det(D-M)} = 1?$$

Atiyah-Singer index appears

Taking the axial U(1) anomaly into account,

$$\frac{\det(D+M)}{\det(D-M)} = \frac{\det(D-M)}{\det(D-M)} \times \exp\left(i\pi \underbrace{\frac{1}{32\pi^2}\int d^4x FF}_{=I} \right) = (-1)^I.$$

I =Atiyah-Singer index

Our proposal: Why don't we use massive Dirac operator to "define" the index?

(We use anomaly rather than symmetry.)

“New” Atiyah-Singer index

$$\frac{\det(D+M)}{\det(D-M)} = \frac{\det i\gamma_5(D+M)}{\det i\gamma_5(D-M)} = \frac{\prod_{\lambda_{+M}} i\lambda_{+M}}{\prod_{\lambda_{-M}} i\lambda_{-M}} = \exp\left[\frac{i\pi}{2}\left(\sum_{\lambda_{+M}} \mathrm{sgn}\lambda_{+M} - \sum_{\lambda_{-M}} \mathrm{sgn}\lambda_{-M}\right)\right]$$

$\lambda_{\pm M}$: eigenvalues of $\gamma_5(D \pm M)$.

$$I = \frac{1}{2}\left[\eta(\gamma_5(D+M)) - \eta(\gamma_5(D-M))\right].$$

$$\equiv \frac{1}{2}\eta(\gamma_5(D+M))^{reg.}$$

$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

Good: chirality is not important.

Bad: not written by zero modes only.

Contents

More physical set-up?

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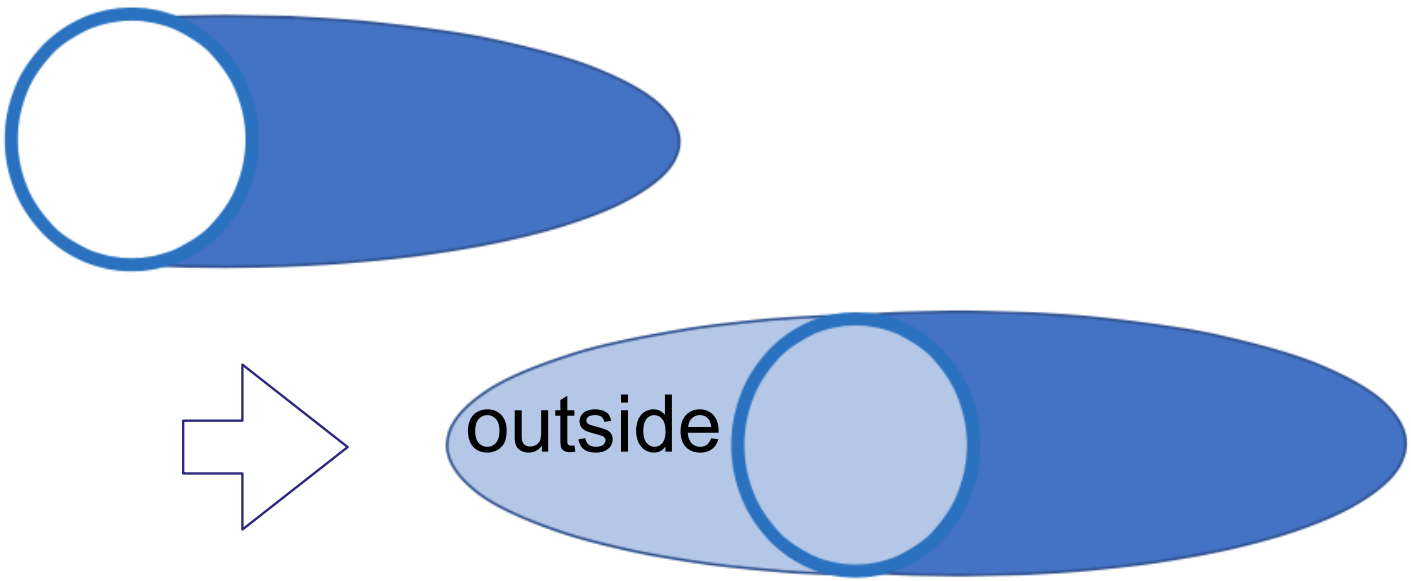
In physics,

1. Any boundary has “outside”:

   ~~manifold + boundary~~ → domain-wall.



More physical set-up?

In physics,

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More physical set-up?

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   → but automatically chosen.

4. Edge-localized modes play the key role.

Domain-wall Dirac operator

[Jackiw-Rebbi 1976, Callan-Harvery 1985, Kaplan 1992 …]

Let us consider

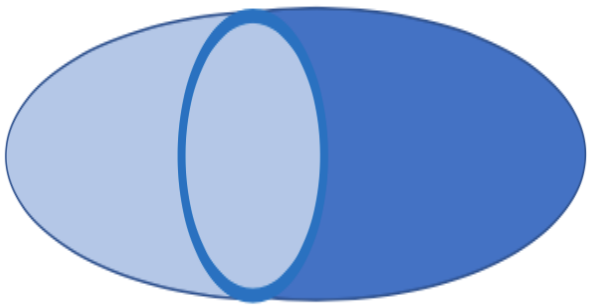
$$D_{4\mathrm{D}} + M\epsilon(x_4), \quad \epsilon(x_4) = \mathrm{sgn}x_4$$

on a closed manifold

with sign flipping mass,

without assuming any boundary condition

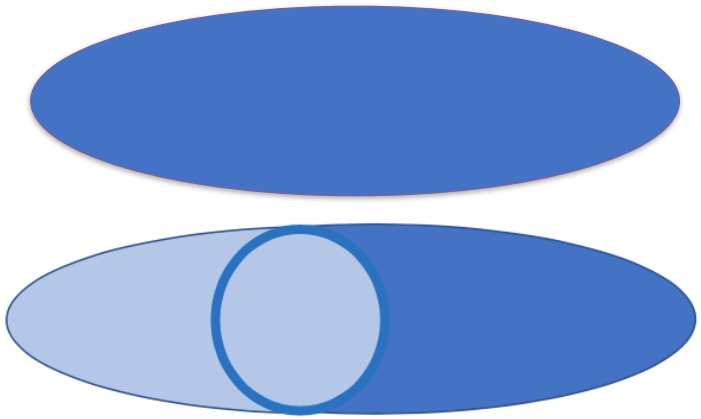
(we expect it dynamically given.).



Here our "domain-wall fermion" is in 4D continuum. (not 5D lattice)

“new” APS index [F-Onogi-Yamaguchi 2017]

$$\frac{1}{2}\eta(\gamma_5(D+M))^{reg} = \text{AS index}$$



$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

which can be shown by Fujikawa-method.

Fujikawa method:

$$\frac{1}{2}\eta(H_{DW}) = \frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D + M\varepsilon(x_4))}{\sqrt{\{\gamma_5(D + M\varepsilon(x_4))\}^2}}$$

1. choose regularization
2. choose complete set to evaluate trace
3. perturbation

Fujikawa method:

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1. choose regularization

   Pauli-Villars: $-\frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D - M_2)}{\sqrt{\{\gamma_5(D - M_2)\}^2}}$ $\quad M_2 \gg M$

2. choose complete set to evaluate trace

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Fujikawa method:

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2. choose complete set to evaluate trace

   eigen set of $\{\gamma_5(D^{\mathrm{free}} + M\varepsilon(x_4))\}^2$

3. perturbation

Complete set in the free case

Solutions to

$$\{\gamma_5(D^{\rm free} + M\varepsilon(x_4))\}^2\phi = \left[-\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4)\right]\phi = \lambda^2\phi$$

are $\varphi(x_4)\otimes e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ where

$$\varphi^{\omega}_{\pm,o}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega\mp M)e^{i\omega|x_4|} + (i\omega\pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\rm edge}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|},$$

Edge mode appears !

Here, $\omega = \sqrt{\boldsymbol{p}^2 + M^2 - \lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,{\rm edge}}_{\pm,e/o} = \pm\varphi^{\omega,{\rm edge}}_{\pm,e/o}$

3D direction = conventional plane waves.

“Automatic” boundary condition

We didn’t put any boundary condition by hand. But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\Big|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the domain-wall.

This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

Bulk & edge contributions

By a simple perturbation, we obtain

$$\frac{1}{2}\eta(H_{DW})^{bulk} = \frac{1}{2}\sum_{bulkmodes}(\phi^{bulk})^\dagger \mathrm{sgn}(H_{DW})\phi^{bulk} = \frac{1}{64\pi^2}\int d^4x \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}(x) + O(1/M).$$

$$\frac{1}{2}\eta(H_{DW})^{edge} = \frac{1}{2}\sum_{edgemodes}\phi^{edge}(x)^\dagger \mathrm{sgn}(H_{DW})\phi^{edge}(x) = -\frac{1}{2}\eta(iD^{3\mathrm{D}})|_{x_4=0}$$

$$-\frac{1}{2}\eta(H_{PV}) = \frac{1}{64\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}(x) + O(1/M).$$

$$\Rightarrow$$

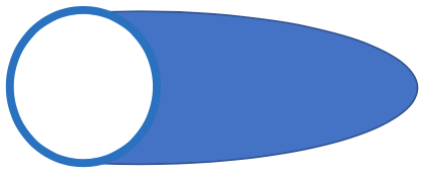
$$\frac{1}{2}\eta(\gamma_5(D + M\epsilon(x_4)))^{reg} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

Contents

Just a coincidence?

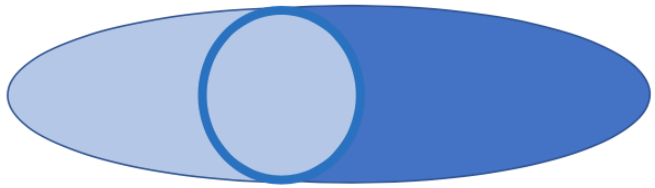
$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H^{reg}_{DW})$$

on general even-dimensional manifolds ?



APS

1. massless Dirac (even in bulk)
2. non-local boundary cond.
   (depending on gauge fields)
3. SO(3) rotational sym. on boundary is lost.
4. no edge mode appears.
5. manifold + boundary



Domain-wall fermion

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(3) rotational sym. on boundary is kept.
4. Edge mode describes eta-invariant.
5. closed manifold + domain-wall

Mathematician's response

In 2018, I gave a talk in a workshop

organized by Mikio Furuta (U. Tokyo).

He said "Mathematicians should give a general proof."

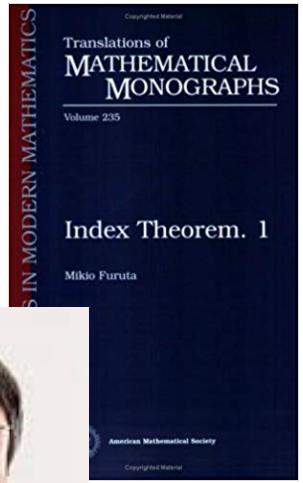
Moreover, only 1 week later,

he proposed a sketch of proof for

$$\frac{1}{2}\eta(H^{reg}_{DW}) = Ind(D_{\mathrm{APS}})$$

[F, Furuta, Matsuo, Onogi,

Yamaguchi, and Yamashita, arXiv:1910.01987]



Theorem

(F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019)

For any APS index of a massless Dirac operator on a even-dim. curved manifold X+ with boundary, there exists a massive (domain-wall) Dirac operator on a closed manifold, sharing its half with X+, and its eta invariant is equal to the original index.

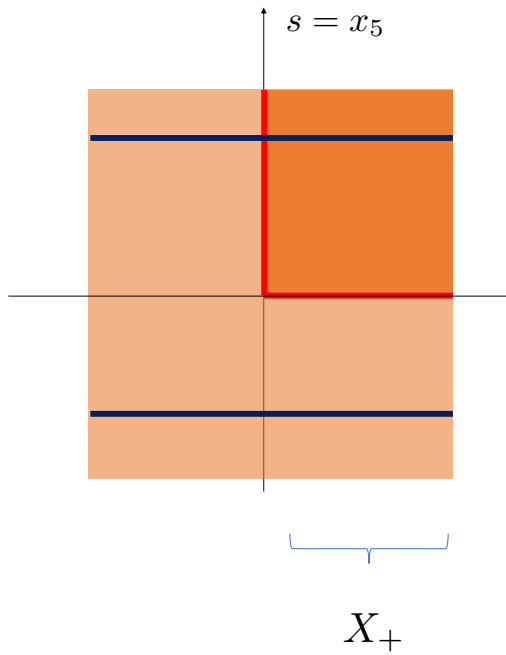
Sketch of the proof

(F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019)

We introduce an extra dimension and consider a Dirac operator on the higher dim. manifold.

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_4, x_5)) \\ -\partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_4, x_5)) & 0 \end{pmatrix}$$

$$m(x_4, x_5) = \begin{cases} M & \text{for } x_4 > 0 \ \& \ x_5 > 0 \\ 0 & \text{for } x_4 = 0 \ \& \ x_5 = 0 \\ -M_2 & \text{otherwise} \end{cases}$$



With 2 different evaluations, we can show

$$Ind(D^{5D}) = Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

Contents

✔ 1. Introduction

APS index is physicist-unfriendly.

✔ 2. Massive Dirac operator index without boundary

$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

✔ 3. New index with ~~boundary~~ domainwall [FOY 2017]

$\mathfrak{I} = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

✔ 4. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

$Ind(D_{\mathrm{APS}})$ and $\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ are different expressions of the same 5D Dirac index.

5. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

6. Summary

Atiyah-Singer index on a lattice

Overlap fermion action $S = \sum_x \bar{q}(x) D_{ov} q(x)$

[Neuberger 1998]

is invariant under

$$q \to e^{i\alpha\gamma_5(1-aD_{ov})} q, \quad \bar{q} \to \bar{q} e^{i\alpha\gamma_5}.$$

[Luescher 1998]

but fermion measure transforms

as $Dq\bar{q} \to \exp\left[2i\alpha \mathrm{Tr}(\gamma_5 + \gamma_5(1 - aD_{ov}))/2\right] Dq\bar{q}$

which reproduces U(1)A anomaly.

Moreover, $\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$ is AS index !

[Hasenfratz et al. 1998]

On the lattice, AS is O.K. but APS is not.

Atiyah-Singer index can be formulated by overlap Dirac operator,

but APS was not known.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$$

1. Lattice version of APS condition impossible, as it does not have a form $N + B$

2. Any boundary condition breaks GW relation [Luescher 2006].

Cf. Kikukawa, “Suri-kagaku” 2020 Jan.

But the lattice AS index theorem "knew" the eta invariant!

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \qquad D_{ov} = \frac{1}{a}\left(1 + \gamma_5\frac{H_W}{\sqrt{H_W^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} \quad = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

$$H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

Cf. Itoh-Iwasaki-Yoshie 1982, Adams 2001

The lattice index theorem "knew"

1. index can be given with massive Dirac.
2. chiral symmetry is not important.

   Wilson Dirac operator is enough.

Unification of index theorems

index theorems with massless Dirac

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1-aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

index theorems with massive Dirac

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))?$

YES ! F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675

APS index on a lattice

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675

On 4-dimensional Euclidean lattice with periodic boundaries (T^4), we have shown

$$-\frac{1}{2}\eta(H_{\rm DW}) = \frac{1}{32\pi^2}\int_{0<x_4<L_4} d^4x\epsilon^{\mu\nu\rho\sigma}{\rm tr}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}\eta(iD^{\rm 3D})|_{x_4=0} + \frac{1}{2}\eta(iD^{\rm 3D})|_{x_4=L_4}, \quad +O(a)$$

$$\varepsilon(x_4) = {\rm sgn}(x_4 - a/2){\rm sgn}(T - x_4 - a/2)$$

* Bulk part is similar to that of AS index [H.Suzuki 1998].

Note that LHS is always an integer.

Contents

Summary

Massive (domain-wall) fermion is physicist-friendly:

APS index can be formulated (even on a lattice).

Moreover, it is mathematically rich:

The eta inv. of massive Dirac unifies the index theorems.

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$