Multi-Soliton Dynamics of Anti-Self-Dual Gauge Fields

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Based on the work

Masashi Hamanaka, SCH [arXiv:2106.01353].
Cf1: M. Hamanaka, SCH, JHEP 2010 (2020) 101. [arXiv:2004.09248].
Cf2: C.R. Gilson, M. Hamanaka, SCH, J.C.C.Nimmo J. Physics A, 53 (2020) 404002. [arXiv:2004.01718].

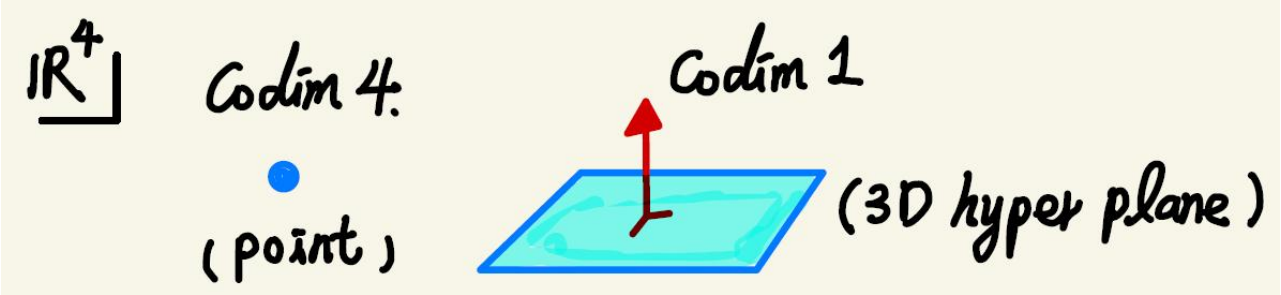
Introduction

The anti-self-dual Yang-Mills (ASDYM) equations

- Important in QFT, geometry, and integrable systems.
- Many lower-dimensional integrable equations (e.g. KdV eq) can be derived from ASDYM by the dimensional reduction (Ward 1985).
- For the split signature $(+,+,-,-)$, the ASDYM equations are the EOM of effective action for $\mathrm{N}=2$ string theories (Ooguri-Vafa 1991).

What is the soliton in this talk :

- Not instanton (codimension 4 type soliton)
- But codimension 1 type soliton (soliton wall).



Introduction

KdV 1-Soliton in (1+1)-dimensional integrable systems

$$u(x,t) = 2\kappa^2 \mathrm{sech}^2 X, \quad X := \kappa x + \kappa^3 t + \delta$$

with energy density $2u^3 - (u_x)^2 = 16\kappa^6 \left(2\mathrm{sech}^6 X - \mathrm{sech}^4 X\right)$.

ASDYM 1-Soliton on 4D spaces (Hamanaka-SCH 2020 [arXiv:2106.01353].)

(We consider the action density $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$ simply as an analogue of energy density.)

- $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \propto \left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right)$, X : linear function of x^1, x^2, x^3, x^4.
- $G = \mathrm{SU}(2)$ for split signature $(+,+,-,-)$

ASDYM Multi-Soliton (Cf: KdV Multi-soliton is well-known)

- Does the multi-soliton exist ?
- Can the gauge group be unitary ?
- Is it possible to have some applications to $\mathrm{N} = 2$ string theories ?

General theory of the anti-self-dual Yang-Mills equations

The complex representation of **ASDYM** ($G = \mathrm{GL}(\mathrm{N}, \mathbb{C})$)

$$F_{zw} = 0,\ F_{\widetilde{z}\widetilde{w}} = 0,\ F_{z\widetilde{z}} - F_{w\widetilde{w}} = 0$$

$\Updownarrow$ equivalent expression

Yang equation (Yang 1977 $\mathrm{G} = \mathrm{SU}(2)$, Brihaye-Fairlie-Nuyts-Yates 1978 $\mathrm{G} = \mathrm{SU}(\mathrm{N})$)

$$\partial_{\widetilde{z}}[(\partial_z J)J^{-1}] - \partial_{\widetilde{z}}[(\partial_z J)J^{-1}] = 0, \quad J : N \times N \ \text{(Yang's } J\text{-matrix)}$$

$$\Downarrow\ J = \widetilde{h}^{-1} h$$

ASD Gauge Fields

$$A_z = h^{-1}(\partial_z h),\ A_w = h^{-1}(\partial_w h),\ A_{\widetilde{z}} = \widetilde{h}^{-1}(\partial_{\widetilde{z}}\widetilde{h}),\ A_{\widetilde{w}} = \widetilde{h}^{-1}(\partial_{\widetilde{w}}\widetilde{h})$$

$\Downarrow$ imposing reduction conditions on $(z, \widetilde{z}, w, \widetilde{w})$

Gauge fields A_μ for $(+,+,-,-)$. $\begin{pmatrix} z & w \\ \widetilde{w} & \widetilde{z} \end{pmatrix} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -(x^2 - x^4) & x^1 - x^3 \end{pmatrix}$

ASD Yang-Mills equations and Darboux transforamtion

Lax representation of ASDYM (Nimmo-Gilson-Ohta 2000)

$$(*)\left\{\begin{array}{l} L(\phi) := [\partial_w - (\partial_w J)J^{-1}]\phi - (\partial_{\widetilde{z}}\phi)\zeta = 0 \\ M(\phi) := [\partial_z - (\partial_z J)J^{-1}]\phi - (\partial_{\widetilde{w}}\phi)\zeta = 0 \end{array}\right. , \quad \zeta\text{: } N \times N \text{ constant matrix}$$

(ϕ is the general solution of $(*)$ *w.r.t.* the spectral parameter ζ.)

The ASDYM eq (Yang eq) can be derived from $(*)$ by the condition $L(M(\phi)) - M(L(\phi)) = 0$.

Darboux Transformation (Nimmo-Gilson-Ohta 2000)

$$\widetilde{\phi} = \phi\zeta - \psi\Lambda\psi^{-1}\phi, \quad \widetilde{J} = -\psi\Lambda\psi^{-1}J$$

(ψ is a specified solution of $(*)$ *w.r.t.* a specific spectral parameter Λ.)

$\Longrightarrow$ $(*)$ is form invariant under the Darboux transformation :

$$\left\{\begin{array}{l} \widetilde{L}(\widetilde{\phi}) := [\partial_w - (\partial_w \widetilde{J})\widetilde{J}^{-1}]\widetilde{\phi} - (\partial_{\widetilde{z}}\widetilde{\phi})\zeta = 0 \\ \widetilde{M}(\widetilde{\phi}) := [\partial_z - (\partial_z \widetilde{J})\widetilde{J}^{-1}]\widetilde{\phi} - (\partial_{\widetilde{w}}\widetilde{\phi})\zeta = 0 \end{array}\right.$$

$$\Longrightarrow \quad \overbrace{J_1}^{\textbf{Seed solution}} \xrightarrow{\textbf{Dar}} J_2 \xrightarrow{\textbf{Dar}} J_3 \xrightarrow{\textbf{Dar}} J_4 \xrightarrow{\textbf{Dar}} \dots \xrightarrow{\textbf{Dar}} J_{n+1} \xrightarrow{\textbf{Dar}} \dots$$

ASD Yang-Mills equations and Darboux transforamtion

After n iterations of the Darboux transformation, the J-matrix can be written in terms of the quasideterminant (noncommutative version of determinant) :

$$J_{n+1} = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{0} \end{vmatrix} J_1, \quad \begin{cases} \psi_i, \Lambda_i, J_{n+1} : N \times N \\ \psi_i : \text{specified solutions of } (*) \\ \qquad w.r.t.\ \Lambda_i \end{cases}$$

(Gilson-Hamanaka-SCH-Nimmo 2020, $G = \mathrm{GL(N,\mathbb{C})}$ [arXiv:2004.01718])

Quasideterminant : (Gelfand-Retakh 1991)

$$\begin{vmatrix} A_{nN\times nN} & B_{nN\times N} \\ C_{N\times nN} & \boxed{D_{N\times N}} \end{vmatrix} = D - CA^{-1}B \ : N \times N$$

ASDYM 1-Soliton Solution $(G = \mathrm{SU}(2),\ (+,+,-,-))$

Seed Solution J_1

$$\text{Set } J_1 = I_{2\times 2} \quad \overset{\text{solve}}{\Longrightarrow} \quad (*) \begin{cases} L(\phi) = (\partial_w)\phi - (\partial_{\tilde{z}}\phi)\zeta = 0 \\ M(\phi) = (\partial_z)\phi - (\partial_{\tilde{w}}\phi)\zeta = 0 \end{cases}$$

$\Downarrow$ 1 iteration of the Darboux transformation

Nontrivial Solution J_2

$$J_2 = \begin{vmatrix} \psi & 1 \\ \psi\Lambda & \boxed{0} \end{vmatrix} = -\psi\Lambda\psi^{-1}, \quad \text{where} \begin{cases} \psi = \begin{pmatrix} ae^{L} & \overline{b}e^{-\overline{L}} \\ -be^{-L} & \overline{a}e^{\overline{L}} \end{pmatrix}, \ a, b, \in \mathbb{C} \\ \text{is a specified solution of } (*) \ \textit{w.r.t.} \\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}, \ \lambda \in \mathbb{C} \end{cases}$$

$$= \frac{-1}{\det(\psi)} \begin{pmatrix} \lambda|a|^2 e^{L+\overline{L}} + \overline{\lambda}|b|^2 e^{-(L+\overline{L})} & (\overline{\lambda} - \lambda)a\overline{b}e^{L-\overline{L}} \\ (\overline{\lambda} - \lambda)\overline{a}be^{-(L-\overline{L})} & \overline{\lambda}|a|^2 e^{L+\overline{L}} + \lambda|b|^2 e^{-(L+\overline{L})} \end{pmatrix},$$

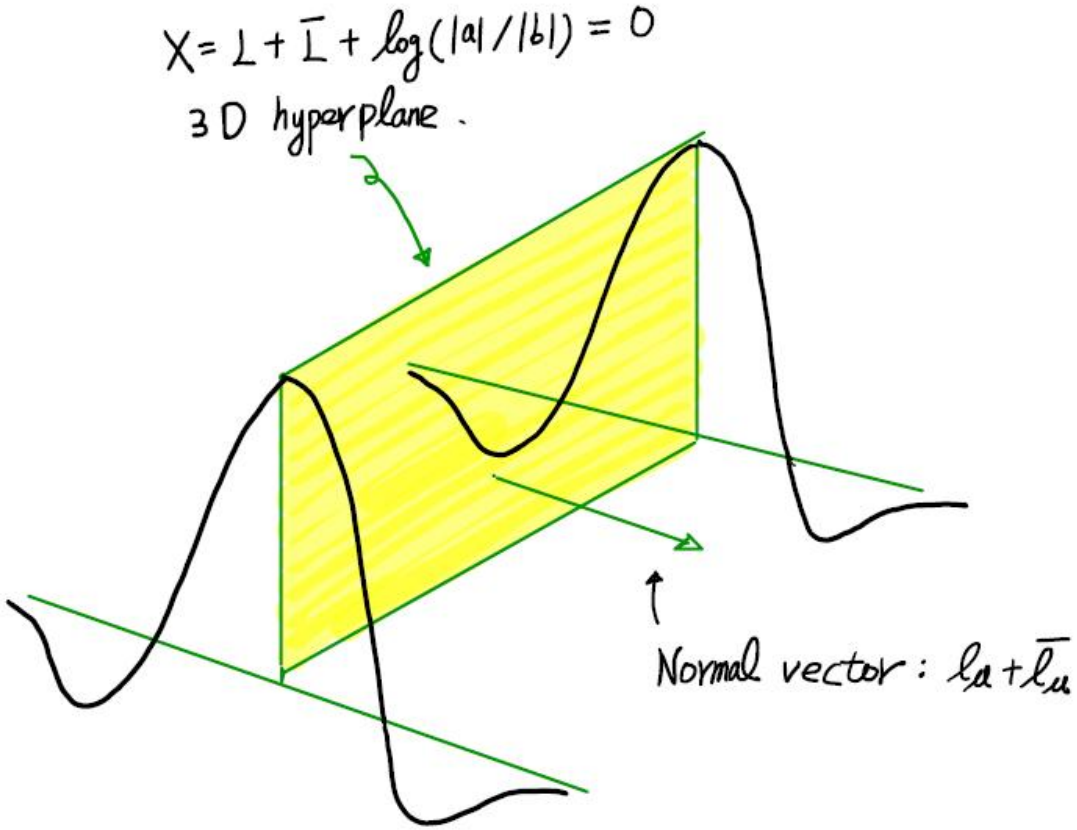
$$L = \frac{1}{\sqrt{2}}\left[(\lambda\alpha + \beta)x^1 + (\lambda\beta - \alpha)x^2 + (\lambda\alpha - \beta)x^3 + (\lambda\beta + \alpha)x^4\right]$$

ASDYM 1-Soliton Solution $(G = \mathrm{SU}(2),\ (+,+,-,-))$

Action Density (Hamanaka-SCH 2020 [arXiv:2004.09248])

$$\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \propto \left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right), X = L + \overline{L} + \log(|a|\,/\,|b|)$$

- Codimension 1 type soliton : (soliton wall)



ASDYM n-Soliton Solution $(G = \mathrm{SU}(2),\ (+,+,-,-))$

Candidate of n-Soliton Solution : (Hamanaka-SCH 2021 [arXiv:2106.01353])

$$J_{n+1} := \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^{n-1} & \psi_2\Lambda_2^{n-1} & \cdots & \psi_n\Lambda_n^{n-1} & 0 \\ \psi_1\Lambda_1^{n} & \psi_2\Lambda_2^{n} & \cdots & \psi_n\Lambda_n^{n} & \boxed{0} \end{vmatrix}, \quad \psi_i = \begin{pmatrix} a_i e^{L_i} & \overline{b}_i e^{-\overline{L}_i} \\ -b_i e^{-L_i} & \overline{a}_i e^{\overline{L}_i} \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \overline{\lambda_i} \end{pmatrix}$$

$$L_i := \frac{1}{\sqrt{2}}\left[(\lambda_i\alpha_i + \beta_i)x^1 + (\lambda_i\beta_i - \alpha_i)x^2 + (\lambda_i\alpha_i - \beta_i)x^3 + (\lambda_i\beta_i + \alpha_i)x^4\right]$$

Question :

Can the **"n-soliton solution"** be interpreted as n **intersecting soliton walls** ?

(The calculation of action density $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ by J_{n+1} is almost impossible.)

ASDYM n-Soliton Solution (Asymptotic behavior)

Consider a comoving frame related to the l-th 1-soliton :

$$J_2^{(l)} = -\psi_n^{(l)} \Lambda_l (\psi_n^{(l)})^{-1}, \quad \psi_n^{(l)} = \begin{pmatrix} a_l e^{L_l} & \overline{b}_l\, e^{-\overline{L}_l} \\ -b_l e^{-L_l} & \overline{a}_l\, e^{\overline{L}_l} \end{pmatrix},$$

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu\ (l)} = 8\left[(\alpha_l \overline{\beta}_l - \overline{\alpha}_l \beta_l)(\lambda_l - \overline{\lambda}_l)\right]^2 \left(2\mathrm{sech}^2 X_l - 3\mathrm{sech}^4 X_l\right).$$

More precisely, we define $r := \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2}$ and consider the asymptotic limit $r \to \infty$ such that

$$\begin{cases} X_l \text{ is a finite real number} \\ X_{i, i\neq l} \ \to \pm\infty \quad (i.e.\ \mathrm{Tr} F_{\mu\nu} F^{\mu\nu (i\neq l)} \to 0) \end{cases}.$$

We want to know the asymptotic behavior of the "n-soliton solution"

$$J_{n+1} \overset{r\to\infty}{\longrightarrow} \ ? \qquad \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \overset{r\to\infty}{\longrightarrow} \ ?$$

ASDYM n-Soliton Solution (Asymptotic behavior)

Consider a comoving frame related to the l-th 1-soliton :

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$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu\ (l)} = 8\left[(\alpha_l \overline{\beta}_l - \overline{\alpha}_l \beta_l)(\lambda_l - \overline{\lambda}_l)\right]^2 \left(2\mathrm{sech}^2 X_l - 3\mathrm{sech}^4 X_l\right).$$

The behavior of the "n-soliton"

$$J_{n+1} \overset{r\to\infty}{\longrightarrow} -\widetilde{\psi}_n^{(l)} \Lambda_l (\widetilde{\psi}_n^{(l)})^{-1} \underbrace{D_n^{(l)}}_{\text{(Constant matrix)}}, \quad \widetilde{\psi}_n^{(l)} = \begin{pmatrix} a'_l e^{L_l} & \overline{b}'_l\, e^{-\overline{L}_l} \\ -b'_l e^{-L_l} & \overline{a}'_l\, e^{\overline{L}_l} \end{pmatrix},$$

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \overset{r\to\infty}{\longrightarrow} 8\left[(\alpha_l \overline{\beta}_l - \overline{\alpha}_l \beta_l)(\lambda_l - \overline{\lambda}_l)\right]^2 \left(2\mathrm{sech}^2 X'_l - 3\mathrm{sech}^4 X'_l\right),$$

$X'_l = X_l + \Delta_l$ (Δ_l : The phase shift of l-th 1-soliton).

Example : 3-Soliton

3-Soliton on 2D space $(x^1 = x,\ x^2 = x^4 = 0,\ x^3 = t)$

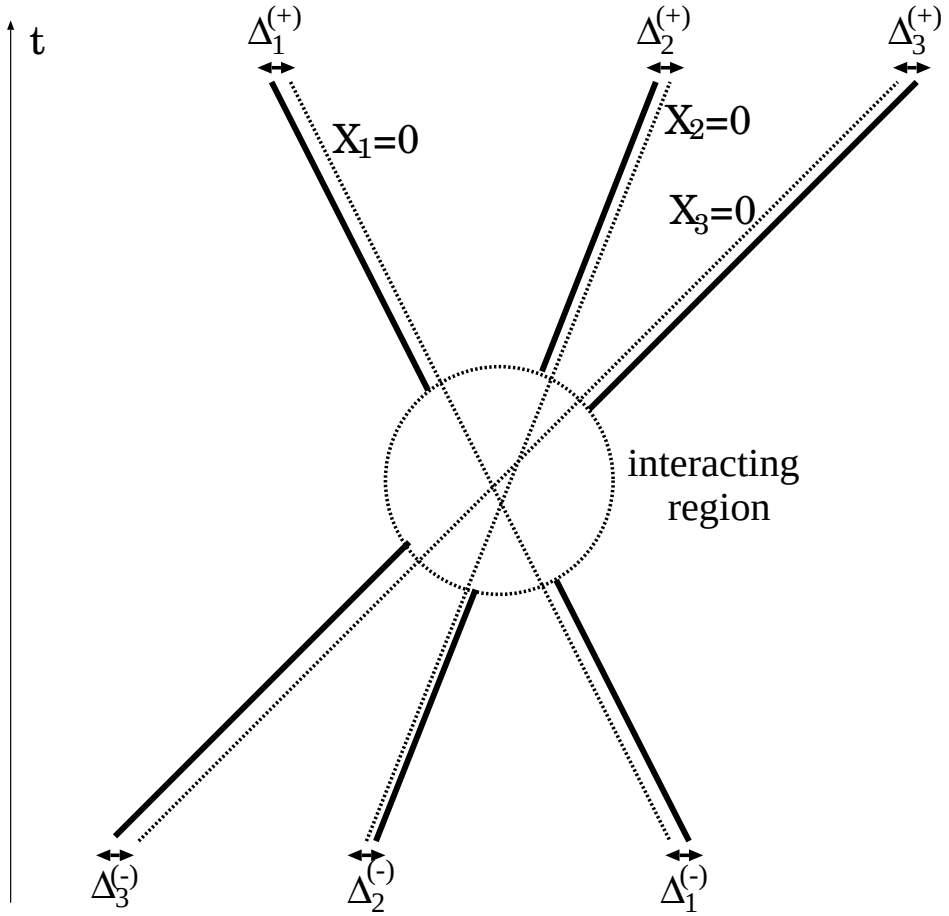
$$\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \sim 2\mathrm{sech}^2 X_I' - 3\mathrm{sech}^4 X_I', I = 1, 2, 3.$$

$$X_I' = \begin{cases} X_I + \Delta_I^{(+)} & \text{when} \quad t \to +\infty \\ X_I + \Delta_I^{(-)} & \text{when} \quad t \to -\infty \end{cases}, \quad \text{In fact, } \Delta_I^{(-)} = -\Delta_I^{(+)}.$$

$$\begin{aligned}
\Delta_1^{(+)} &= -\Delta_1^{(-)} = -\log\left|\frac{\lambda_1 - \lambda_2}{\lambda_1 - \overline{\lambda}_2}\right| - \log\left|\frac{\lambda_1 - \lambda_3}{\lambda_1 - \overline{\lambda}_3}\right|, \\
\Delta_2^{(+)} &= -\Delta_2^{(-)} = +\log\left|\frac{\lambda_2 - \lambda_1}{\lambda_2 - \overline{\lambda}_1}\right| - \log\left|\frac{\lambda_2 - \lambda_3}{\lambda_3 - \overline{\lambda}_3}\right|, \\
\Delta_3^{(+)} &= -\Delta_3^{(-)} = +\log\left|\frac{\lambda_3 - \lambda_1}{\lambda_3 - \overline{\lambda}_1}\right| + \log\left|\frac{\lambda_3 - \lambda_2}{\lambda_3 - \overline{\lambda}_2}\right|.
\end{aligned}$$

Example : 3-Soliton

3-Soliton on 2D space $(x^1 = x,\ x^2 = x^4 = 0,\ x^3 = t)$



Summary and Future Work

Summary

- For any given $I \in \{1, 2, ..., n\}$, the distribution of action density of n-soliton in the asymptotic region

  $$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \overset{r\to\infty}{\longrightarrow} 8\left[(\alpha_I \overline{\beta}_I - \overline{\alpha}_I \beta_I)(\lambda_I - \overline{\lambda}_I)\right]^2 \left(2\mathrm{sech}^2 X_I' - 3\mathrm{sech}^4 X_I'\right)$$

  behaves like the I-th 1-soliton.

  $\Longrightarrow$ The n-soliton can be interpreted as n intersecting soliton walls.

- The gauge fiels A_μ given by the n-solition solution J_{n+1} can be proved to be anti-hermitian and traceless $(\mathrm{G} = \mathrm{SU}(2))$. (Hamanaka-SCH 2021 [arXiv:2106.01353]).

  $\Longrightarrow$ n intersecting "branes" in $\mathrm{N} = 2$ string theories.

Summary and Future Work

Future Work

- **Split signature** $(+,+,-,-)$
  To understand the role and applications of the soliton walls in $\mathrm{N}=2$ string theories.
- **Euclidean signature** $(+,+,+,+)$
  To construct the soliton walls for $G=\mathrm{SU}(2)$.
- **Minkowski singature** $(+,-,-,-)$
  To construct the soliton walls in the Yang-Mills-Higgs theory.
- **Applications to the axisymmetric gravitational field**
  The Ernst equation (axisymmetric Einstein equation) can be derived from the ASDYM equations by dimensional reduction.