E-strings, E_8 Weyl invariant Jacobi forms and Conway invariant Jacobi forms on Leech lattice

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Outline

- Jacobi forms of lattice index
- Weyl invariant E_8 Jacobi forms
- Conway invariant Jacobi forms on the Leech lattice

Jacobi forms of lattice index

Weak Jacobi forms on lattice L with weight k and index $t\langle z,z\rangle$

$\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ satisfies the following transformation laws:

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \exp\left(t\pi i \frac{c\langle z,z\rangle}{c\tau+d}\right)\varphi(\tau,z),$$

$$\varphi(\tau, z+x\tau+y) = \exp\left(-t\pi i[\langle x,x\rangle\tau + 2\langle x,z\rangle]\right)\varphi(\tau,z), \quad x,y \in L,$$

$$\varphi(\tau,z) = \sum_{n=0}^{\infty}\sum_{\ell\in L'} f(n,\ell)e^{2\pi i(n\tau+\langle \ell,z\rangle)}, \qquad L' \text{ is dual of } L.$$

- If $f(n,\ell) = 0$ whenever $2nt - \langle \ell,\ell\rangle < 0$, then it is called a holomorphic Jacobi form.

Elliptic genera of BPS string of 6d SCFTs involve weak Jacobi forms on lattice $L_2 \oplus L_G \oplus L_F$ with index

$$A\epsilon_1^2 + B\epsilon_1\epsilon_2 + C\epsilon_2^2 + t_G\langle m,m\rangle_G + t_F\langle m,m\rangle_F.$$

An E_8 surprise for Weyl invariant Jacobi forms

- Particular interesting when L is the root lattice of simple Lie algebra $\mathfrak{g}$ and φ is $\mathfrak{g}$ Weyl invariant, where t also called level.

Theorem (Wirthmüller 1992)

Weak Weyl-invariant Jacobi forms for $A_r, B_r, C_r, D_r, E_{6,7}, F_4, G_2$ are polynomially generated by $r+1$ generators with non-positive weights.

Theorem (H. Wang 2018)

Weak E_8 Weyl-invariant Jacobi forms are NOT polynomially generated.

An old physics puzzle in (Huang-Klemm-Poretschkin 2013)

- E-string theory is a simple 6d $(1,0)$ SCFT with flavor E_8.
- Comes from M2-branes streched between a M5-brane and a M9-brane in the Horava-Witten picture
- The refined free energies $F_{n,g}$ of t E-strings should be expressed by E_8 Jacobi forms of level t.
- However, the seemingly perfect modular ansatz for $F_{n,g}$ turns out to be inconsistent for $t \geq 5$. Why?

Sakai's nine E_8 "generators" $A_{1,2,3,4,5}, B_{2,3,4,6}$

In the study on the Seiberg-Witten curve of E-strings, (Sakai 2011) defined 9 holomorphic E_8 Jacobi forms. Using Hecke transformation T_i and weight 2 modular forms g_2 on $\Gamma_0(j)$,

$$A_1 := \Theta_{E_8}(\tau, \vec{m}), \quad A_4 := \Theta_{E_8}(\tau, 2\vec{m}), \quad A_i := T_i A_1, \ i = 2, 3, 5,$$
$$B_j := \mathrm{Tr}_{\mathrm{SL}_2(\mathbb{Z})}(g_2(\tau) A_1(j\tau, j\vec{m})), \ j = 2, 3, 4, 6. \quad i, j \text{ are the level}.$$

They reduce to Eisenstein series with E_8 fugacities off:

$$A_i(\vec{m} \to 0) = E_4, \ i = 1, 2, 3, 4, 5, \quad B_j(\vec{m} \to 0) = E_6, \ j = 2, 3, 4, 6.$$

Conjecture (Sakai)

For any weak E_8 Weyl-invariant Jacobi form ϕ, there exist $\mathrm{SL}_2(\mathbb{Z})$ modular form f such that $f\phi \in \mathbb{C}[E_4, E_6, A_i, B_j]$.

Proved by (H. Wang 2018). Now we are interested in finding the minimal f which is important both for establishing a practical structure theorem and for physics use.

A surprising weight 18 level 5 Jacobi form $P_{16,5}$

(Del Zotto, Gu, Huang, Kashani-Poor, Klemm and Lockhart 2017) find the following interesting polynomial $P_{16,5} \in \mathbb{C}[E_6, A_i, B_j]$:

$$864A_1^3A_2 + 3825A_1B_2^2 - 770A_3B_2E_6 - 840A_2B_3E_6 + 60A_1B_4E_6 + 21A_5E_6^2$$

By numerical tests, they conjecture (i)

$$\frac{P_{16,5}}{E_4} \text{ is holomorphic!}$$

By numerical search, they also conjecture (ii) there is no polynomial divided by E_6 is still holomorphic, (iii) no polynomial other than $P_{16,5}$ divided by E_4 is still holomorphic.
All these conjectures are proved in (KS-Wang 2021), which play an important role in the final structure theorem.

Space of Weyl invariant Jacobi forms

At fixed level t, there exist $r(t)$ number of generators by which the space of E_8 Weyl invariant weak Jacobi forms $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ are spanned with coefficients in $\mathbb{C}[E_4, E_6]$. Same for the holomorphic case $J^{W(E_8)}_{*,E_8,t}$.

$$\begin{aligned}\sum_{t=0}^{\infty} r(t)x^t &= \frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2(1-x^5)(1-x^6)} \\ &= 1 + x + 3x^2 + 5x^3 + 10x^4 + 15x^5 + 27x^6 + 39x^7 + 63x^8 \\ &\quad + 90x^9 + 135x^{10} + 187x^{11} + 270x^{12} + 364x^{13} + O(x^{14}).\end{aligned}$$

In the modular bootstrap whenever level t E_8 symmetry is involved, we need the explicit level t weak generators.

Remark

$r(t)$ is also the number of conjugacy classes in L^{E_8}/tL^{E_8}.

Structure theorem

Theorem (KS-Wang 2021)

1. *$P_{16,5}/E_4$ is holomorphic and lies in $J^{W(E_8)}_{12,E_8,5}$.*
2. *For $P \in \mathbb{C}[E_6, A_i, B_j]$, if P/E_4 is holomorphic then*

$$P/P_{16,5} \in \mathbb{C}[E_6, A_i, B_j].$$

3. *Every $\varphi_t \in J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ can be expressed uniquely as*

$$\frac{\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j}}{\Delta^{N_t} E_4^{t_1}}, \qquad N_t = \begin{cases} 5t_0, & \text{if } t = 6t_0 \text{ or } 6t_0+1, \\ 5t_0+1, & \text{if } t = 6t_0+2, \\ 5t_0+2, & \text{if } t = 6t_0+3, \\ 5t_0+3, & \text{if } t = 6t_0+4 \text{ or } 6t_0+5. \end{cases}$$

Here t_1 is the integer part of $t/5$, $P_{t_1} \in \mathbb{C}[E_4, E_6, A_i, B_j]$ and $P_j \in \mathbb{C}[E_6, A_i, B_j]$ for $0 \le j < t_1$.

Examples

At level 2, the space $J^{\mathrm{w},W(E_8)}_{*,E_8,2}$ is spanned by three generators

$$\phi_{-4,2} = \frac{A_1^2 - A_2E_4}{\Delta}, \quad \phi_{-2,2} = \frac{A_2E_6 - B_2E_4}{\Delta}, \quad \phi_{0,2} = \frac{A_1^2E_4 - B_2E_6}{\Delta}.$$

At level 3, the space $J^{\mathrm{w},W(E_8)}_{*,E_8,3}$ is spanned by five generators

$$\phi_{-8,3} = \frac{1}{\Delta^2}(6A_1^3E_4 - 9A_1A_2E_4^2 + A_3(3E_4^3 - 10E_6^2) + 30A_1B_2E_6 - 20B_3E_4E_6),$$
$$\phi_{-6,3} = \frac{1}{\Delta^2}(6A_1^3E_6 + 3A_1E_4(10B_2E_4 - 3A_2E_6) - E_4^2(20B_3E_4 + 7A_3E_6)),$$
$$\phi_{-4,3} = \frac{1}{\Delta}(A_1A_2 - A_3E_4),\ \phi_{-2,3} = \frac{1}{\Delta}(A_1B_2 - A_3E_6),\ \phi_{0,3} = \frac{1}{\Delta}(A_1^3 - B_3E_6).$$

- We determine up to level $t \leq 13$, in which case the rank is 364 and the minimal weight of generators is -52.
- These are the genuine bases for modular bootstrap with E_8.

The Leech lattice and the Conway group

- Leech lattice Λ is the unique positive definite even unimodular lattice without root in dimension 24.
- Used in proving the monstrous moonshine conjecture, the densest sphere packing in $\mathbb{R}^{24}$, among many other applications
- Its automorphism group is the Conway group Co_0.
- The quotient of $\mathrm{Co}_0 / \{\pm 1\}$ gives a sporadic simple group Co_1 of order $4,157,776,806,543,360,000$.
- Leech vectors under Co_0 action form Conway orbits.
- For orbit $\mathrm{orb}(v)$, $\frac{1}{2}(v,v)$ is called type.
- Conway orbits up to type 16 described in $\mathbb{ATLAS}$: $p = 65520$

$O_{2,3p}$	$O_{3,256p}$	$O_{4,6075p}$	$O_{5,70656p}$
$O_{6a,518400p}$	$O_{6b,6900p}$	$O_{7,2861568p}$	$O_{8a,3p}$
$O_{8b,12295800p}$	$O_{8c,141312p}$	$O_{9a,12441600p}$	$O_{9b,32972800p}$
$O_{10a,143078400p}$	$O_{10b,279450p}$	$O_{10c,1430784p}$	$\cdots$

Conway invariant Jacobi forms on the Leech lattice

We are interested the Jacobi forms with 24 elliptic variables $\mathfrak{z}$ associated to Λ, which are invariant under Co_0.
For index $t = 1$, the space of both weak and holomorphic Jacobi forms of such type is spanned by the Theta function of Λ

$$A_1(\tau, \mathfrak{z}) = \Theta_\Lambda(\tau, \mathfrak{z}) = 1 + O_2 q^2 + O_3 q^3 + O_4 q^4 + O_5 q^5 + (O_{6a} + O_{6b}) q^6 + O_7 q^7 + \dots$$

Its reduction to weight 12 modular form is

$$\theta_\Lambda(\tau) = E_{12}(\tau) - \frac{65520}{691}\Delta(\tau) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}(\sigma_{11}(n) - \tau(n))q^n$$
$$= 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + 4629381120q^5 + \dots$$

Structure theorem for index 2 and 3

Theorem (KS-Wang 2021)

As free modules over $M_*(\mathrm{SL}_2(\mathbb{Z}))$,

1. $J^{\mathrm{w},\mathrm{Co}_0}_{*,\Lambda,2}$ *is generated by four forms of weights* $-4, -2, 0, 0$.
2. $J^{\mathrm{Co}_0}_{*,\Lambda,2}$ *is generated by four forms* A_2, $\Phi_{12,2}$, B_2 *and* HB_2 *of weights* $12, 12, 14, 16$.
3. $J^{\mathrm{w},\mathrm{Co}_0}_{*,\Lambda,3}$ *is generated by ten forms of weights* $-14, -12, -12, -12, -10, -8, -6, -4, -2, 0$.
4. $J^{\mathrm{Co}_0}_{*,\Lambda,3}$ *is generated by ten forms* A_3, $\Phi_{12,3}$, $\Psi_{12,3}$, B_3, $\Psi_{14,3}$, HB_3, $H\Psi_{14,3}$, $\Psi_{16,3}$, H^2B_3 *and* $H^2\Psi_{14,3}$ *of weights* $12, 12, 12, 14, 14, 16, 16, 16, 18, 18$.

Here $\Phi_{12,m}$ are the mth Fourier coefficient of the famous Borcherds' automorphic form Φ_{12}:

$$\Phi_{12}(Z) = \Delta(\tau) \cdot \exp\Big(- \sum_{m=1}^{\infty} (T_m(\Delta^{-1}A_1))(\tau, \mathfrak{z}) e^{2\pi i m \omega} \Big)$$

Applications of Conway invariant Jacobi forms

- Decomposition of many products of Conway orbits

$$\mathrm{orb}(r) = \sum_{v \in \mathrm{Co}_0 \cdot r} e^{2\pi i \langle v, \mathfrak{z} \rangle}, \quad \mathrm{orb}(v)\,\mathrm{orb}(u) = \sum_{r \in \Lambda} c_r \,\mathrm{orb}(r).$$

  Here is the smallest example: $O_2 \otimes O_2 = 196560 O_0 \oplus 4600 O_2 \oplus 552 O_3 \oplus 46 O_4 \oplus 2 O_5 \oplus 2 O_{6b} \oplus O_{8a}$. Note each side has about 3.8×10^{11} terms, which would be extremely hard to calculate in a brutal way.
- Modular linear differential equations
- Conjugate classes of orbits modulo $t\Lambda$
- Intersection between Conway orbits and Leech vectors by the pullback to Eichler–Zaiger Jacobi forms

Questions

From (Borcherds 1985),

$$|\Lambda/\Lambda| = 1, \quad |\Lambda/2\Lambda| = 4, \quad |\Lambda/3\Lambda| = 10, \quad |\Lambda/4\Lambda| = 31, \quad |\Lambda/t\Lambda| = ?$$

We know the $t = 1$ case corresponding to a Monster CFT with $c = 24$ from compactification on a $\mathbb{Z}_2$ orbifold of the torus $\mathbb{R}^{24}/\Lambda$ (Dixon-Ginsparg-Harvey 1988). It is intriuging to ask whether there exist

- a level 2 Monster CFT such that it has 4 characters with q expansion coefficients as dimensions of Monster reps?
- a level 3 Monster CFT such that it has 10 characters?
- ...

Characters should be related to Conway invariant Jacobi forms.

- Recall $(E_8)_n$ WZW models have 1,3,5,10 characters for $n = 1, 2, 3, 4$.

Besides, the $t = 4$ representative system of Conway orbits are recently found in my second paper

Thank you for listening!

K. Sun, H. Wang
Weyl invariant E_8 Jacobi forms and E-strings.
arXiv: 2109.10578

K. Sun, H. Wang
Conway invariant Jacobi forms on the Leech lattice.
arXiv: 2111.10999