

Basis decompositions of genus-one string integrals

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Based on 2112.**** with Oliver Schlotterer and Carlos Rodriguez

EAJS 2021

Nov. 25

Motivation

- String amplitudes are derived from moduli-space integrals
- Simplified correlators became important:
 - 1) computational efficiency,
 - 2) primarily to unravel elegant structures and relations of string amplitudes and their field-theory limit

e.g. KK [Kleiss, Kuijf, 89'] and BCJ relations [Bern, Carrasco, Johansson, 08'] at tree level
[Stieberger et al, 13',14']

KLT relations [Kawai, Lewellen, Tye, 86'], $\text{GR} = \text{YM}^2$ [Mafra, Schlotterer, Stieberger, Carrasco, Henrik,...]
Reduce Einstein-Yang-Mills tree amplitudes to gauge-theory ones [Schlotterer, 16'][see
Song's talk]

- The variety of tree-level insights motivate the quest for analogous reductions for loop-level correlators

Review: string integrand at tree level

- Any n-point massless open string tree-level amplitude takes the form

$$A^{\text{tree}} = \int_{z_1 < \dots < z_n} \frac{d^n z}{\text{SL}(2)} \overbrace{\prod_{i < j} z_{ij}^{-s_{ij}}}^{\text{KN}} I_n \left(\{k_i, \epsilon_i, z_i\} \right)$$

with $s_{ij} := -\frac{\alpha'}{2} k_i \cdot k_j$, $z_{ij} = z_i - z_j$,

$\text{SL}(2) \rightarrow$ e.g. $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$

- There may be terms in I_n that are proportional to a single PT factor

$$\text{PT}(1, 2, \dots, n) := \frac{1}{z_{12} z_{23} \cdots z_{n1}}$$

KK and BCJ relations at tree level

- Algebraically, by repeating using $\frac{1}{z_i z_j} = \frac{1}{z_i - z_j} \left(\frac{1}{z_i} - \frac{1}{z_j} \right)$, we have

$$\text{PT}(1,2,3,4) + \text{PT}(2,1,3,4) + \text{PT}(2,3,1,4) = 0 \quad \rightarrow \quad \text{KK relations}$$

- On the support of integration-by-parts, we have

$$\int \text{KN} \text{PT}(1234) = \int \text{KN} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} \text{PT}(1324) \quad \rightarrow \quad \text{BCJ relations}$$

- These explain KK and BCJ relations, $(n-1)! \rightarrow (n-2)! \rightarrow (n-3)!$

$$\text{PT}(1, \rho(2,3,\dots,n-2), n-1, n)$$

General basis decomposition at tree level

- In general, there would be a product of shorter PT factors in I_n , e.g

$$\text{PT}(1,2,\dots,m) \text{ PT}(m+1,m+2,\dots,n)$$

- A general formula to break a shorter PT factor [Schlotterer, 16']

$$\text{PT}(12\dots m)(\dots) \stackrel{\text{IBP}}{=} \frac{1}{1 + s_{12\dots m}} \left(\sum_{\ell=2}^m \sum_{j=m+1}^{n-1} \sum_{\rho \in \{2,3,\dots,\ell-1\} \sqcup \{n,n-1,\dots,\ell+1\}} (-1)^{n-\ell+1} \frac{s_{\ell j}}{z_{1\rho_1} z_{\rho_1 \rho_2} \cdots z_{\rho_{|\rho|} \ell} z_{\ell j}} \right) (\dots)$$

e.g. $\underbrace{\frac{1}{z_{12}^2}}_{-\text{PT}(12)} (\dots) \stackrel{\text{IBP}}{=} \frac{1}{(1 + s_{12}) z_{12}} \left(\frac{s_{23}}{z_{23}} + \frac{s_{24}}{z_{24}} + \dots \right) (\dots)$

Similar for $\text{PT}(123) = 1/(z_{12} z_{23} z_{31})$

General basis decomposition at tree level

$$\bullet \text{PT}(12\cdots m)(\dots) \stackrel{\text{IBP}}{=} \frac{1}{1 + s_{12\cdots m}} \left(\sum_{\ell=2}^m \sum_{j=m+1}^{n-1} \sum_{\rho \in \{2,3,\cdots,\ell-1\} \sqcup \{n,n-1,\cdots,\ell+1\}} (-1)^{n-\ell+1} \frac{s_{\ell j}}{z_{1\rho_1} z_{\rho_1 \rho_2} \cdots z_{\rho_{|\rho|} \ell} z_{\ell j}} \right) (\dots)$$

For a product of more shorter PT factors, one just needs to use the above identity recursively [Song et al, 18', 19'].

- Once all small PT factors are broken, the integrands can easily further expanded onto $(n-3)!$ BCJ basis.
- Much easier to read the leading order of such string integral, which corresponds to the field theory limit and is closely related to Cachazo-He-Yuan formulas [see Song's talk]
- Our essential task is to find the analog of the above equations at one-loop level

String integrals at genus one

Doubly-periodic functions

- Genus-one strings

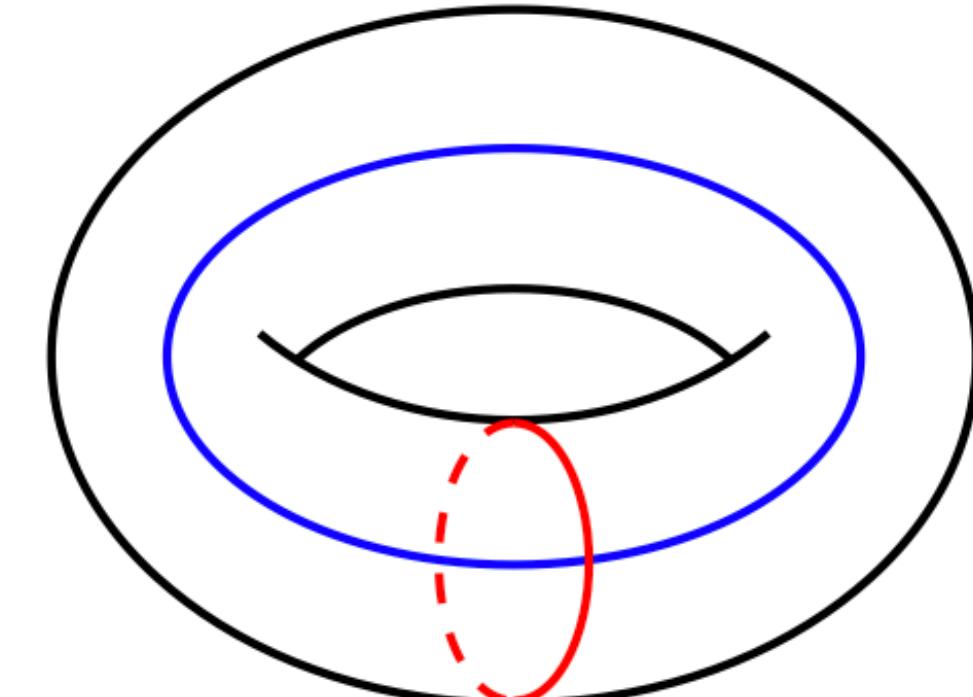
$$\frac{1}{z_i - z_j} \rightarrow \text{doubly-periodic functions on torus, } F(z+1) = F(z+\tau) = F(z)$$

- Generating function: doubly-periodic Kronecker-Eisenstein series

$$\Omega(z, \eta, \tau) := \exp\left(2\pi i \eta \frac{\operatorname{Im} z}{\operatorname{Im} \tau}\right) \frac{\theta'(0, \tau) \theta(z + \eta, \tau)}{\theta(z, \tau) \theta(\eta, \tau)}$$

$$= \sum_{w=0}^{\infty} \eta^{w-1} f^{(w)}(z, \tau)$$

$$\text{e.g. } f^{(0)} = 1, f^{(1)}(z, \tau) = \partial_z \log \theta(z, \tau) + 2\pi i \frac{\operatorname{Im} z}{\operatorname{Im} \tau}$$



String integrals at genus one

Fay identities

- $f^{(k)}$ appear in the string integrand.

For example, the OPE of two Kac-Moody currents on torus reads

$$\langle J^{a_1}(z_1) J^{a_2}(z_2) \rangle \sim 2f_{12}^{(2)} - (f_{12}^{(1)})^2 + (\text{free of } z)$$

where $f_{ij}^{(k)} := f^{(k)}(z_{ij}, \tau)$. At tree level, this OPE gives $\sim \text{PT}(12)$.

- Fay identity, $f_{12}^{(1)} f_{23}^{(1)} + f_{13}^{(2)} + \text{cyc}(123) = 0$

$$\text{In general } \Omega(z_1, \eta_1, \tau) \Omega(z_2, \eta_2, \tau) = \Omega(z_1, \eta_1 + \eta_2, \tau) \Omega(z_2 - z_1, \eta_2, \tau) + (1 \leftrightarrow 2)$$

$$\text{Remind } \frac{1}{z_i z_j} = \frac{1}{z_i - z_j} \left(\frac{1}{z_j} - \frac{1}{z_i} \right)$$

- Variant $\Omega(z, \eta, \tau) \Omega(-z, \xi, \tau) = \Omega_{\pi\eta}(z, \eta - \xi, \tau) (\hat{g}^{(1)}(\xi, \tau) - \hat{g}^{(1)}(\eta, \tau)) + \partial_z \Omega(z, \eta - \xi, \tau)$
where $\hat{g}^{(1)}(\eta, \tau) = \partial_\eta \log \theta(\eta, \tau) + \frac{\pi\eta}{\text{Im } \tau}$

String integrals at genus one

IBP at one loop

- The massless n-point one-loop amplitudes of the open string give rise to integrals of the form ($z_1 = 0$)

$$\int_{\mathcal{C}^{(*)}} \left(\prod_{j=2}^n dz_j \right) \overline{f_{i_1 j_1}^{(k_1)} f_{i_2 j_2}^{(k_2)} \dots} \overbrace{\exp \left(\sum_{i < j} s_{ij} G(z_{ij}, \tau) \right)}^{\text{KN}^\tau}$$

with different integration domains $\mathcal{C}(^*)$ for the cylinder and the Möbius strip.

- The bosonic Green function satisfies $\partial_{z_i} G(z_{ij}, \tau) = -f_{ij}^{(1)}$ and therefore

$$\partial_{z_i} \text{KN}^\tau = -\text{KN}^\tau \sum_{j \neq i}^n s_{ij} f_{ij}^{(1)}$$

- Remind $\partial_{z_i} \text{KN} = -\text{KN} \sum_{j \neq i} \frac{s_{ij}}{z_{ij}}$

Chains and cycles

- Conjectured chain basis for I_n^τ [Mafra, Schlotterer, 19']:

$$\Omega_{12\dots n} := \Omega_{12}(\eta_{23\dots n}) \Omega_{23}(\eta_{3\dots n}) \dots \Omega_{n-1,n}(\eta_n) ,$$

and its relabelling over $2,3,\dots,n$.

Here $\Omega_{ij}(\eta_{ab\dots c}) := \Omega(z_{ij}, \eta_{ab\dots c}, \tau)$, $\eta_{23\dots n} = \eta_2 + \eta_3 + \dots + \eta_n$.

- Obstacles: cycles

$$C_{(12\dots m)} := \Omega_{1,2}(\eta_{2\dots m,1}) \Omega_{2,3}(\eta_{3\dots m,1}) \dots \Omega_{m-1,m}(\eta_{m,1}) \Omega_{m,1}(\eta_1)$$

$$\text{e.g. } C_{(12)} = \Omega_{1,2}(\eta_{2,1}) \Omega_{2,1}(\eta_1)$$

- We have to use Fay identities and IBP to break all these cycles.

Length-2 cycle

- Using the variants of Fay identities, we have

$$C_{(12)} := \Omega(z_{1,2}, \eta_{2,1})\Omega(z_{2,1}, \eta_1) = \Omega_{1,2}(\eta_2) \left(\hat{g}^{(1)}(\eta_2, \tau) - \hat{g}^{(1)}(\eta_{21}, \tau) \right) + \partial_{z_1}\Omega_{12}(\eta_2)$$

- On the support of IBP,

$$\left(\partial_{z_1}\Omega_{12}(\eta_2) \right) \mathbf{KN}^\tau \stackrel{\text{IBP}}{=} -\Omega_{12}(\eta_2) \left(\partial_{z_1} \mathbf{KN}^\tau \right) = \Omega_{12}(\eta_2) \mathbf{KN}^\tau s_{12} f_{12}^{(1)}$$

- Together with $\partial_z\Omega(z, \eta, \tau) - \partial_\eta\Omega(z, \eta, \tau) = (\hat{g}^{(1)}(\eta, \tau) - f^{(1)}(z, \tau)) \Omega(z, \eta, \tau)$ one can derive

$$C_{(12)} \cong \frac{1}{1 + s_{12}} \left(s_{12} \partial_{\eta_2} - \hat{g}_1(\eta_2) + (1 + s_{12}) \tilde{V}_1(\eta_2, \eta_1) \right) \Omega_{1,2}(\eta_2)$$

where $\tilde{V}_1(\eta_I, \eta_J) := \hat{g}_1(\eta_I) + \hat{g}_1(\eta_J) - \hat{g}_1(\eta_{I,J})$.

Length-3 cycle

- The first non-trivial example is to break

$$C_{(123)} := \Omega_{1,2}(\eta_{2,3,1})\Omega_{2,3}(\eta_{3,1})\Omega_{3,1}(\eta_1).$$

- We found

$$\begin{aligned} & (1 + s_{123})C_{(123)} \\ & \cong \left((s_{13} + s_{23})\partial_{\eta_3} - s_{23}\partial_{\eta_2} - \hat{g}_1(\eta_3) - s_{12}\tilde{V}_1(\eta_3, \eta_2) + (1 + s_{1,2,3})\tilde{V}_1(\eta_3, \eta_1) \right) \Omega_{1,2,3} \\ & - \left((s_{12} + s_{23})\partial_{\eta_2} - s_{23}\partial_{\eta_3} - \hat{g}_1(\eta_2) - s_{13}\tilde{V}_1(\eta_2, \eta_3) + (1 + s_{1,2,3})\tilde{V}_1(\eta_2, \eta_{3,1}) \right) \Omega_{1,3,2} \end{aligned}$$

Note that $\Omega_{1,2,3} := \Omega_{1,2}(\eta_{23})\Omega_{2,3}(\eta_3)$.

Arbitrary cycle

- We even found a formula to break a cycle of arbitrary multiplicity,

$$\begin{aligned}
 & (1 + s_{12\dots n}) C_{(12\dots n)} \\
 & \cong \sum_{\ell=2}^n \sum_{\rho \in \{2,3,\dots,\ell-1\} \sqcup \{n,n-1,\dots,\ell+1\}} (-1)^{n-\ell-1} \left[\sum_{i=1}^n s_{i,\ell} \partial_{\eta_\ell} - \sum_{i=2}^n s_{i,\ell} \partial_{\eta_i} - \hat{g}_1(\eta_\ell) \right. \\
 & \quad \left. + (1 + s_{12\dots n}) \tilde{V}_1(\eta_\ell, \eta_{\ell+1}, \dots, n, 1) - \sum_{i=2}^{\ell-1} S_{i,\rho} \tilde{V}_1(\eta_\ell, \eta_{i,i+1}, \dots, \ell-1) - \sum_{i=\ell+1}^n S_{i,\rho} \tilde{V}_1(\eta_\ell, \eta_{\ell+1}, \ell+2, \dots, i) \right] \Omega_{1,\rho,\ell} \\
 & + \sum_{1 \leq p < u < v < w < q \leq n+1} \sum_{\substack{\rho \in \{2,3,\dots,p\} \sqcup \{n,n-1,\dots,q\} \\ \gamma \in \{p+1,\dots,u-1\} \sqcup \{v-1,\dots,u+1\} \\ \pi \in \{v+1,\dots,w-1\} \sqcup \{q-1,\dots,w+1\}}} \sum_{\sigma \in \{\gamma,u\} \sqcup \{\pi,w\}} (-1)^{n+u+v+w} \left(\sum_{i=q}^n s_{vi} + \sum_{i=1}^p s_{vi} \right) \\
 & \quad \times \left(\hat{g}_1(\eta_{u+1}, \dots, w-1) - \hat{g}_1(\eta_{u+1}, \dots, w) - \hat{g}_1(\eta_{u}, \dots, w-1) + \hat{g}_1(\eta_{u}, \dots, w) \right) \Omega_{1,\rho,v,\sigma}
 \end{aligned}$$

where $S_{i,\rho} := s_{i,1} + \sum_{\substack{2 \leq j \leq n \\ j \text{ precedes } i \text{ in } \rho}} s_{i,j}$

Summary

- In this work we have considerably extended the integration-by-parts technology for one-loop string integrals.
- For a product of cycles, use the single-cycle formula repeatedly.
- Further confirmed the conjectured chain basis at one loop
- Valid for closed strings. Similar formulas hold in the chiral splitting formalism [\[D'Hoker et al, 88', 89'\]](#).

Outlook

- Simplifies one-loop amplitudes in heterotic and bosonic string theories
- Offers a particularly efficient approach to their α' -expansions [\[D'Hoker et al, 15', 16'\]](#). Anomaly problem at 6pt.
- Hoped to yield kinematic factors with a field-theory interpretation or to even signal double-copy structures
- Makes one-loop Cachazo-He-Yuan formulas [\[Geyer, Mason, Cachazo, Song,...\]](#) alive. Produces quadratic propagators [\[Edison et al, 21'\]](#).
- Massive. Two loops.