

Worldsheet Variables for Cluster Configuration Spaces

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Yihong Wang, PZ, to appear**

Motivations

Worldsheet picture for cluster configuration spaces

- Defined by the “u-equations”, encode the factorization property of string integrals
- A-type = open string moduli space $\mathcal{M}_{0,n}$, $u_{i,j} = \frac{z_{i,j-1} z_{i-1,j}}{z_{i,j} z_{i-1,j-1}}$, $z_{i,j} = z_j - z_i$
- Beyond A-type, we can define the “u-spaces”, but lack worldsheet descriptions

Cluster Alphabets

- (Integrated) Feynman Integrals $\sim \sum \text{Polylog}(\alpha_i)$
- Bhabha scattering: $A_3 = \{z_1, z_2, z_3, 1 + z_1, 1 + z_2, 1 + z_3, z_1 - z_2, z_2 - z_3, z_1 - z_3\}$
- 6d hexagon with 1 mass:
$$z_i = f(s, t, m_e)$$

$$D_4 = A_3 \cup \{z_4, 1 + z_4, z_{1,4}, z_{2,4}, z_1 + z_3 z_4, z_2 + z_3 z_4, z_1 - z_2 - z_1 z_2 + z_1 z_3 + z_1 z_4 - z_3 z_4\}$$

Systematic derivation for all finite type?

letter of an alphabet

Chicherin, Henn, Papathanasiou

Topology

- Alphabet as hypersurface arrangement complement
- Point count over \mathbb{F}_p tells us the dimension of cohomology, Euler characteristic, etc.
- e.g. $A_2 = \{z_1, z_2, 1 + z_1, 1 + z_2, z_1 - z_2\}$

$$|\mathcal{M}_{A_2}|_{\mathbb{F}_p} = (p-2)(p-3) = p^2 - 5p + 6 \quad \text{characteristic polynomial } \chi(p)$$

↗

of independent $d \log z_{i,j}$

↖

of independent $d \log z_{i,j} \wedge d \log z_{k,l}$

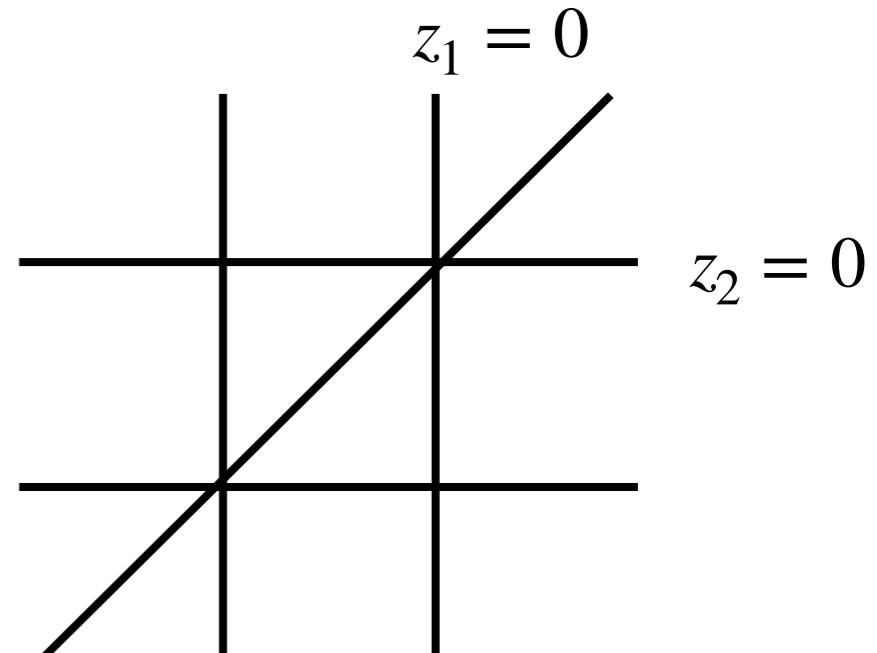
$p = 1$ Euler characteristics

= # of bounded regions in \mathbb{R}^n

= # of solutions to the scattering equations

$$\sum_{1 \leq j < i \leq n} s_{i,j} d \log z_{i,j} = 0$$

Varchenko



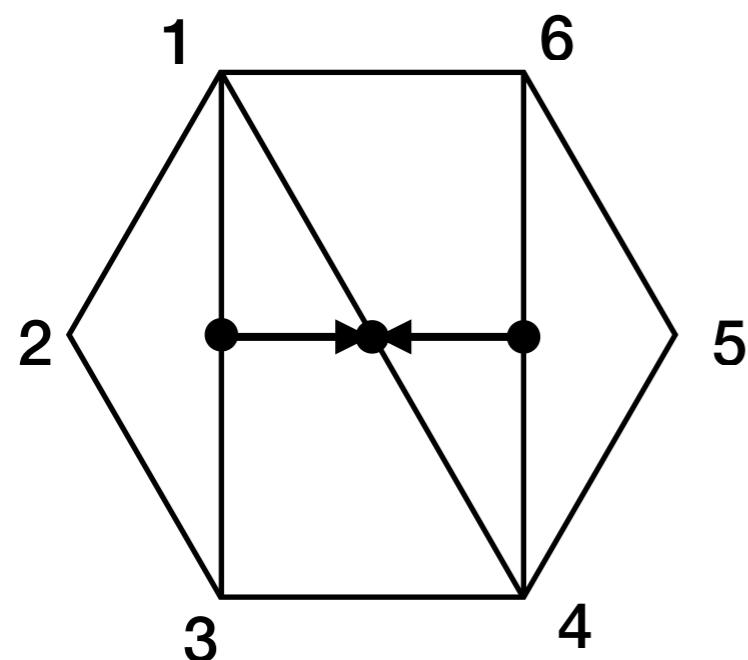
Essential difference: A-type is linear, other types are nonlinear

Koba-Nielsen u-equations

- Moduli space of tree open-string amplitude: n points z_1, \dots, z_n on a disc
- Factorization at massless poles = boundary of the moduli space where some z_i 's collide

$$1 - u_{i,j} = \prod_{(k,l) \text{ incompatible with } (i,j)} u_{k,l}, \quad u_{i,j} = \frac{z_{i,j-1} z_{i-1,j}}{z_{i,j} z_{i-1,j-1}}$$

Each $u_{i,j} \rightarrow 0$ corresponds to a *unique* boundary



$$u_{1,4} \rightarrow 0 \quad \partial A_3 \supset A_1 \times A_1$$

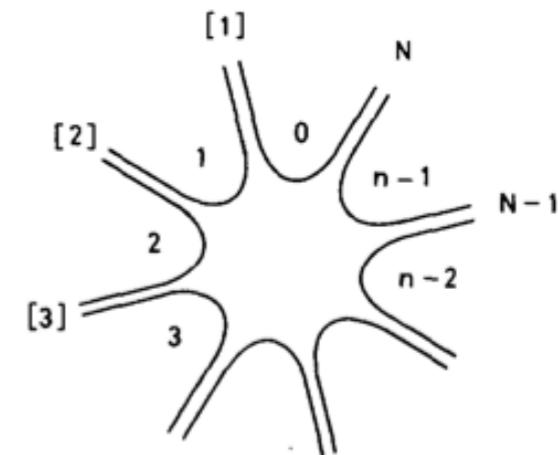
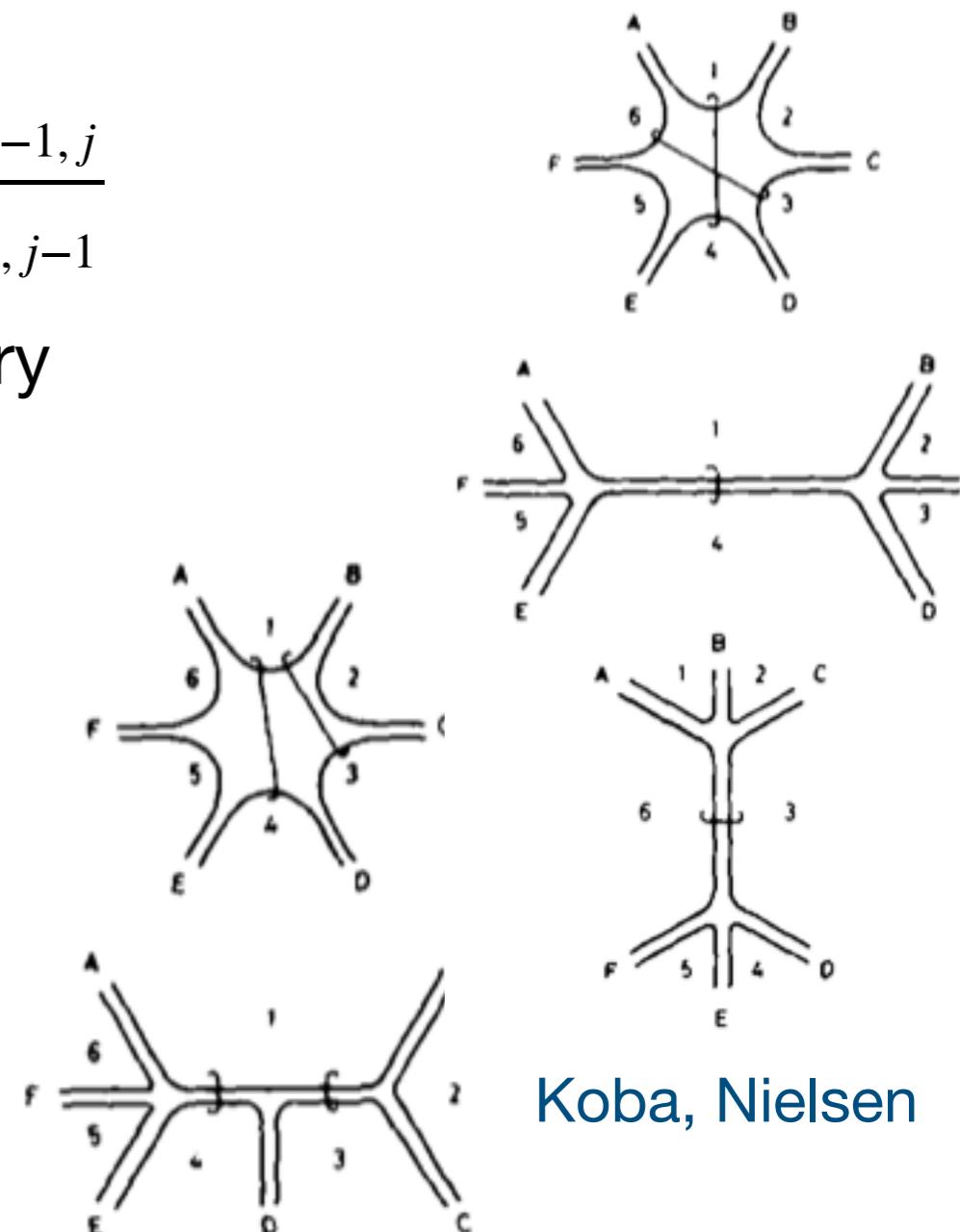


Fig. 2.1. Standard form of the connected quark diagram for an n -meson reaction.



Cluster configuration spaces of finite type

- Space of solutions to the u-equations

$$1 - u_I = \prod_J u_J^{J|I}$$

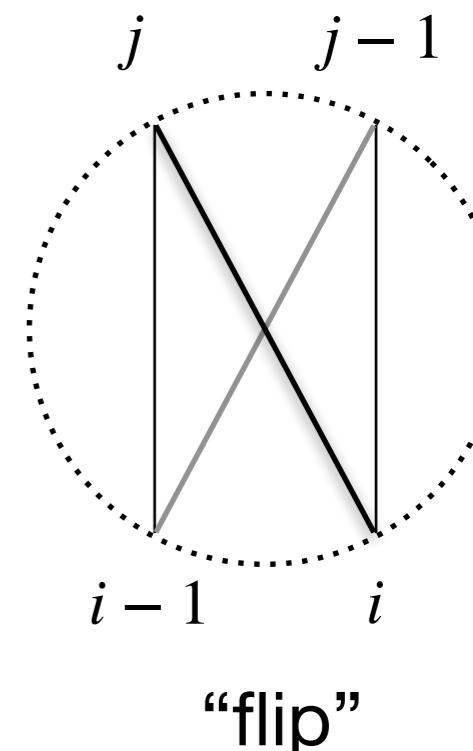
Arkani-Hamed, He, Lam, Thomas

- At each boundary where $u_J \rightarrow 0$, all incompatible $u_I \rightarrow 1$
- We have a *binary* description of the moduli space but do not know what underlying space we are compactifying.
- What is the worldsheet picture for finite-type?
- Local form of the u-equations

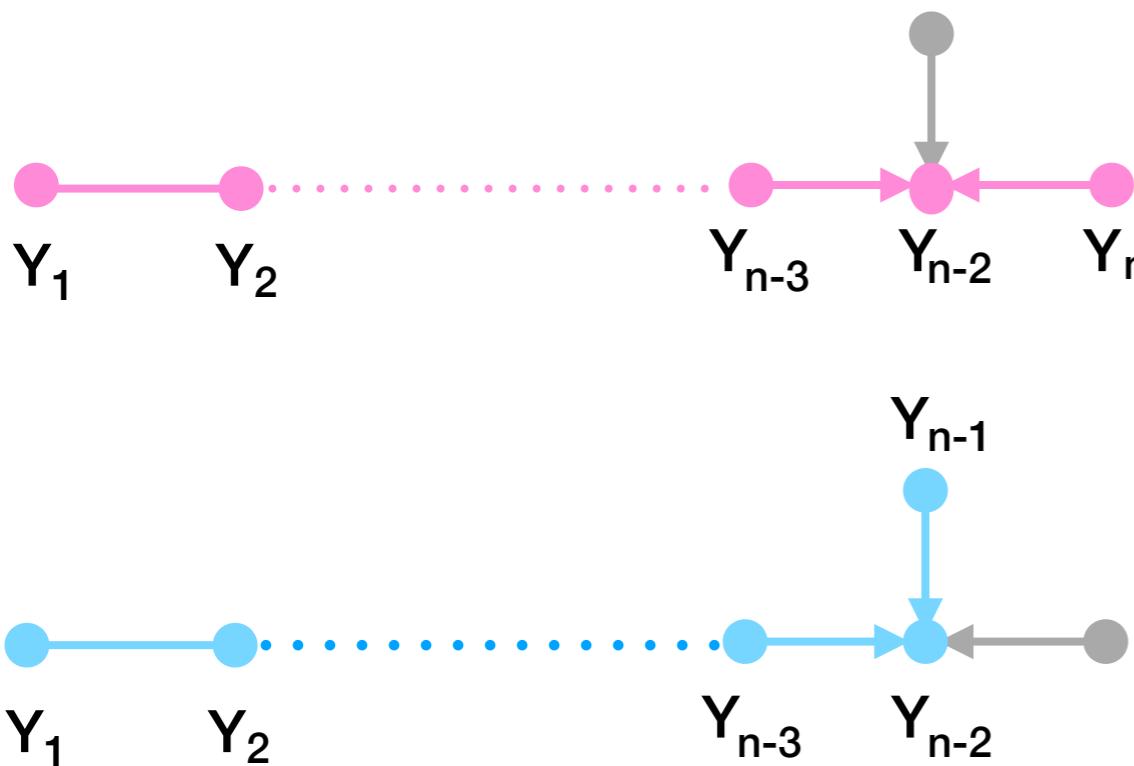
$$\frac{u_{i,j}}{1 - u_{i,j}} \frac{u_{i-1,j-1}}{1 - u_{i-1,j-1}} = \frac{1}{1 - u_{i,j-1}} \frac{1}{1 - u_{i-1,j}}$$

$$Y_{i,j} Y_{i-1,j-1} = (1 + Y_{i-1,j})(1 + Y_{i,j-1}) \quad \text{Zamolodchikov}$$

Y -system can be generalized to finite root systems



The gluing construction of the D_n worldsheet



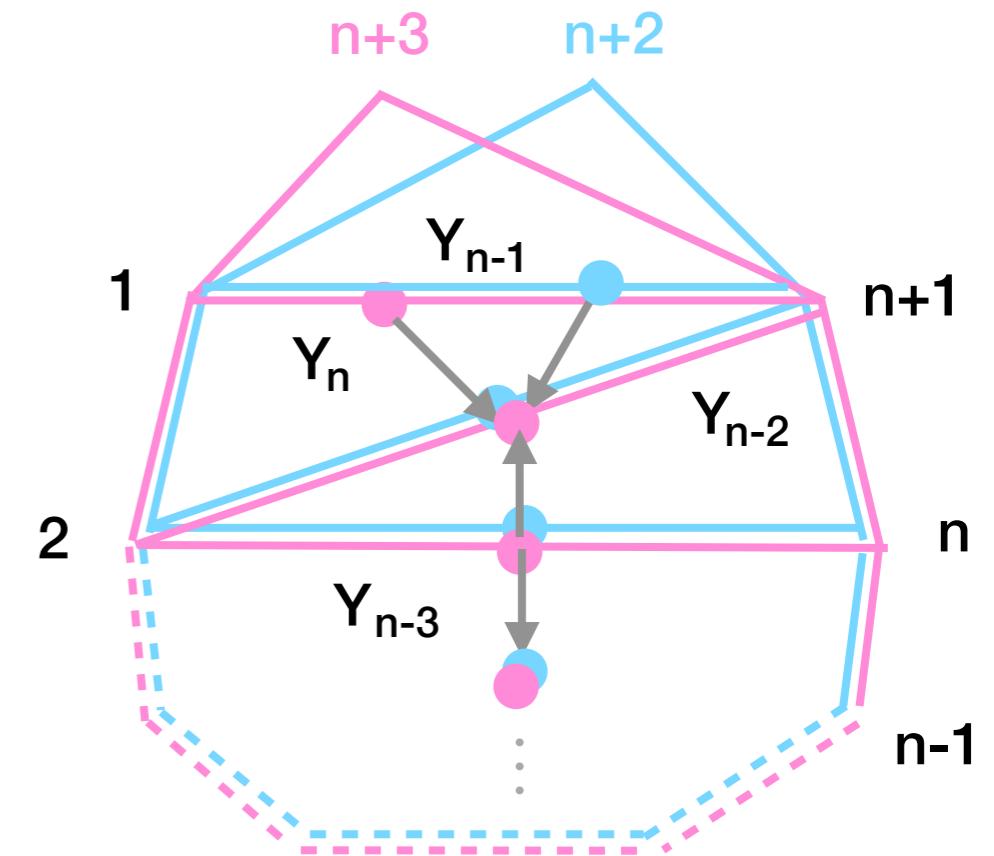
Key observation: $D_n = A_{n-1} \sqcup A_{n-1}$

- Use cross-ratios for initial Y , evolve Y -system to generate others:

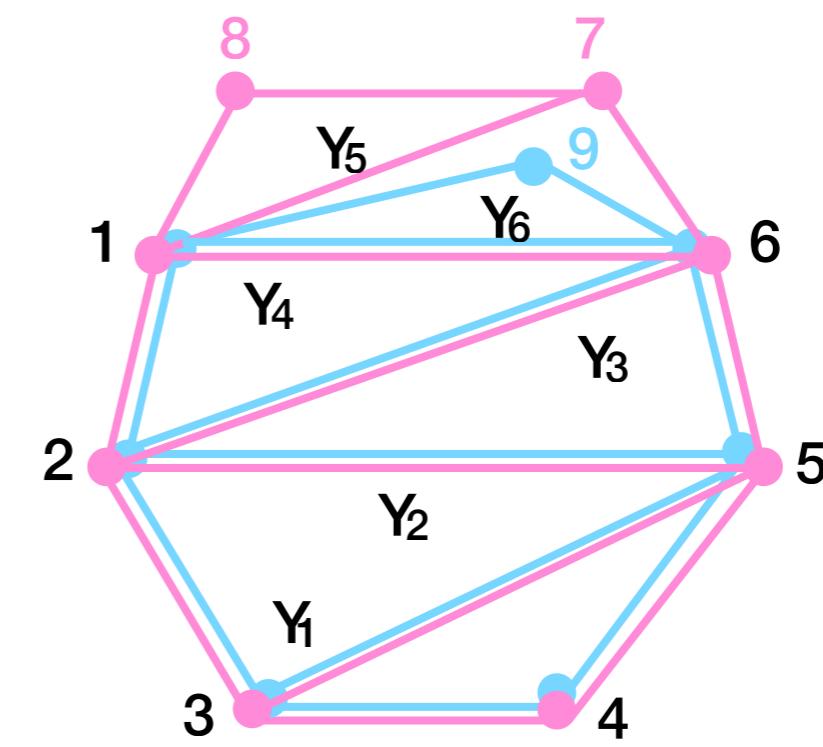
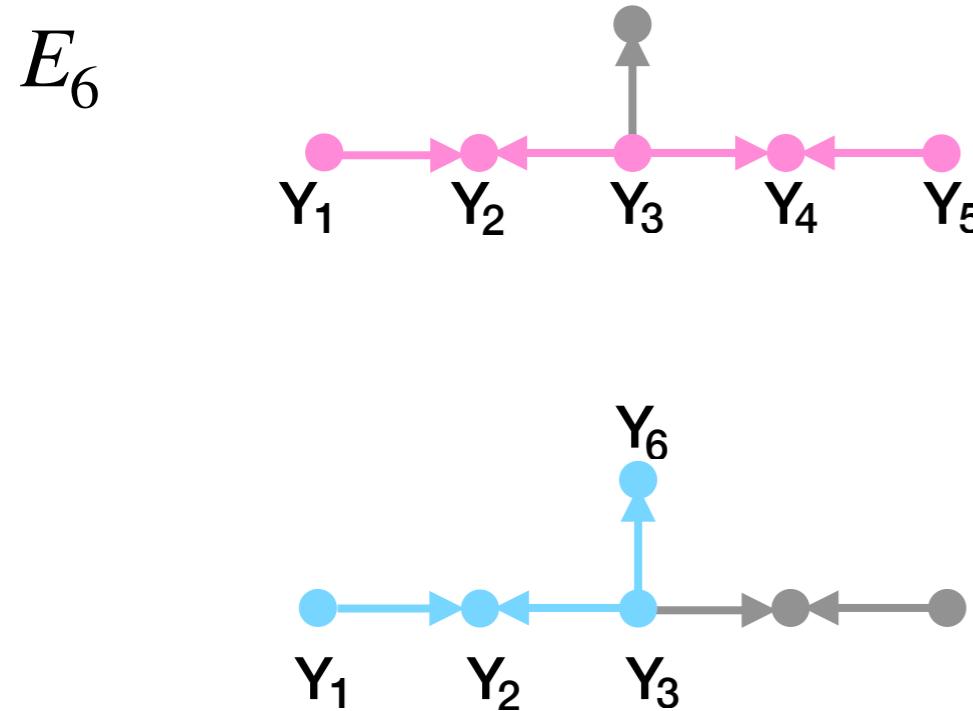
$$u_{i,j} = \frac{z_{i,j-1} z_{i-1,j}}{z_{i,j} z_{i-1,j-1}} \text{ on the **first** sheet}$$

$$u_{i,j} = \frac{w_{i,j-1} w_{i-1,j}}{w_{i,j} w_{i-1,j-1}} \text{ on the **second** sheet}$$

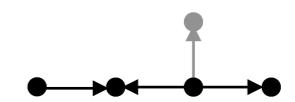
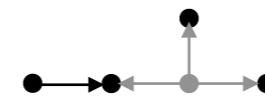
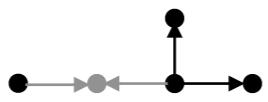
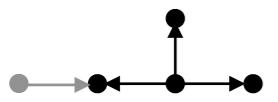
$$u_i = \frac{z_{i,n+3} w_{i-1,i}}{z_{i-1,n+3} w_{i,i}}, \quad u_{\tilde{i}} = \frac{z_{i,n+2} w_{i-1,i}}{z_{i-1,n+2} w_{i,i}}$$



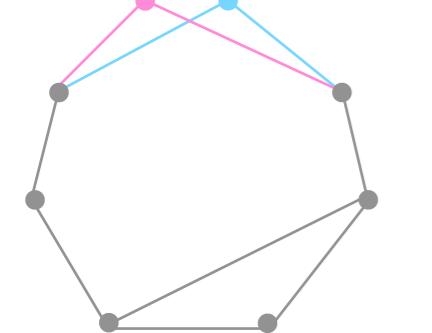
$$\begin{aligned} w_{i,j} &= z_{1,n+3} z_{i,j} z_{n+1,n+2} - z_{1,n+1} z_{i,n+3} z_{j,n+2} \\ &= \det \begin{pmatrix} 1 & 1 & 1 \\ z_i + z_{n+1} & z_1 + z_j & z_{n+2} + z_{n+3} \\ z_i z_{n+1} & z_1 z_j & z_{n+2} z_{n+3} \end{pmatrix} \end{aligned}$$



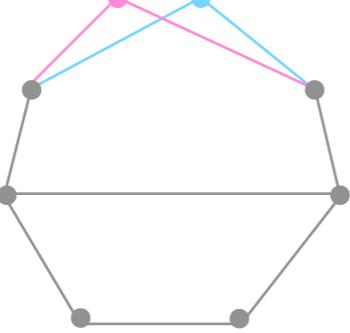
Boundaries = remove nodes of Dynkin diagram



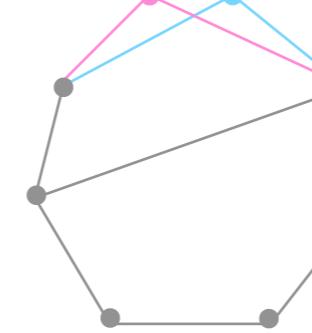
$\partial D_5 \supset$



D_4



$A_1 \times D_3$



$A_2 \times D_2$



$2A_4$

String integrals also factorize this way at the massless poles.

D_n moduli space

$$\mathcal{M}_{D_n} = \left\{ (z_1, z_2, \dots, z_{n+3}) \in (\mathbb{CP}^1)^n \mid \begin{array}{l} z_{i,j}, w_{i,j} \neq 0 \text{ for } 1 \leq i < j \leq n+1 \\ z_{i,n+2}, z_{i,n+3} \neq 0 \text{ for } 1 \leq i \leq n+1 \end{array} \right\} / \widehat{PSL(2, \mathbb{C})}$$

Gauge fix $\{z_1, z_2, z_{n+1}\} = \{-1, 0, \infty\}$: $\{z_{i,j}, w_{i,j}\} \rightarrow$ cluster alphabet of D_n

Gauge fix $\{z_{n+3}, z_1, z_{n+1}\} = \{-1, 0, \infty\}$: $\{z_{i,j}, w_{i,j}\} \rightarrow$ new, simpler alphabet

Count # of points over $\mathbb{F}_p \rightarrow$ Interpolating quasi-polynomials

$$|\mathcal{M}_{D_4}(\mathbb{F}_p)| = p^4 - 16p^3 + 93p^2 - 231p + 207 + \delta_3(p),$$

$$|\mathcal{M}_{D_5}(\mathbb{F}_p)| = p^5 - 25p^4 + 244p^3 - 1156p^2 + 2649p - 2355 + \delta_3(p)(5p - 36) - \delta_4(p).$$

$$\delta_i(p) := \begin{cases} 1 & \text{if } p \equiv 1 \pmod{i}, \\ -1 & \text{otherwise.} \end{cases}$$

D_n Scattering Equations

$$\sum_{1 \leq j < i \leq n+1} s_{i,j} d \log z_{i,j} + \sum_{1 < j < i < n+1} s_{j,i} d \log w_{j,i} + \sum_{1 \leq i \leq n+1} (s_i d \log z_{i,n+3} + \tilde{s}_i d \log z_{i,n+2}) = 0$$

of solutions found numerically by HomotopyContinuation.jl

D_4	D_5	D_6	D_7	D_8
55	674	10215	183256	≥ 3787649

Euler characteristic predicted by point count at $p = 1$

D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
55	674	10215	183256	3787655	88535634	2308393321

Log cohomology

$$|\mathcal{M}_{D_4}(\mathbb{F}_p)| = p^4 - 16p^3 + \boxed{93p^2} - \boxed{231p + 207} + \delta_3(p),$$

$$|\mathcal{M}_{D_5}(\mathbb{F}_p)| = p^5 - 25p^4 + \boxed{244p^3} - \boxed{1156p^2 + 2649p - 2355} + \delta_3(p)(5p - 36) - \delta_4(p).$$

n_2	n_3	2-forms	3-forms	4-forms	5-forms	6-forms
		D_4	$\boxed{93}$	$\boxed{231}$	207	-
		D_5	$\boxed{244}$	$\boxed{1156}$	2649	2355
		D_6	$\boxed{530}$	$\boxed{4070}$	17140	37465
		D_7	$\boxed{1014}$	$\boxed{11460}$	76215	33300
		D_8	$\boxed{1771}$	$\boxed{27629}$		
		D_9	$\boxed{2888}$	$\boxed{59416}$		
		D_{10}	$\boxed{4464}$			

$$\omega_{i,j} = d \log z_{i,j}$$

$$\chi_{i,j} = d \log w_{i,j}$$

Arnold relations

$$\omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j} = 0$$

$$n_2 = \binom{n^2}{2} - \left[\binom{n+2}{3} - n \right] - 3 \binom{n-1}{2} - 2 \binom{n-1}{3}$$

$$= \frac{1}{2}(n-1)(n^3 - n + 2)$$

$$\omega_{i,j} \wedge \chi_{i,j}$$

$$\chi_{i,j} \wedge \chi_{i,k} \& \chi_{i,k} \wedge \chi_{j,k}$$



Agrees with the prediction from point count

Summary

- Worldsheet description of cluster config spaces as cross-ratios of z and w
- New cluster alphabet for D_n and E_6 , (also E_7 and E_8 and BCFG by folding)
- Euler characteristic and log cohomology based on point count

Discussions

- Apply cluster alphabet and dlog relations to bootstrap amplitudes
- Analytic understanding of scattering equations as particles in a 1d potential
- Formulate Knizhnik-Zamolodchikov equation for D_n
- Can our gluing construction for quivers be used in other settings?